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Spaces of positive scalar curvature metrics and parametrised Morse theory

A detection theorem for $d \geqslant 5$ and higher index theory

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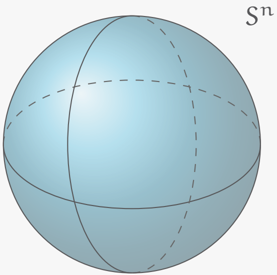
Let (M, g) be a Riemannian manifold.

- Central notion in differential geometry: Riemannian curvature tensor associated to g.
- Tensor contraction turns this into **scalar curvature** $\rightsquigarrow$ smooth function $\mathrm{scal}_g : M \to \mathbb{R}$

Existence question

Given a smooth manifold M, does M admit a metric with $\mathrm{scal}_g > 0$?

- Admitting a **p**ositive **s**calar **c**urvature metric has topological implications
- Only orientable surface admitting psc is S^2
- In fact, there are many (topological) obstructions to admitting psc, e.g. the $\widehat{A}$-genus



$$\mathrm{scal} \equiv \frac{n(n-1)}{r^2}$$

Dimension 2: Gauß–Bonnet

$$0 < \int_M \mathrm{scal}_g \, d\omega = 4\pi \cdot \chi(M)$$

The space of psc metrics

Definition (Space of metrics)

Let $\mathcal{R}(M)$ be the **space of Riemannian metrics** (with the C^∞-topology).

Note: $\mathcal{R}(M)$ is convex, so $\mathcal{R}(M) \simeq *$

Definition (Space of psc metrics)

$$\mathcal{R}^+(M) \coloneqq \{g \in \mathcal{R}(M) \mid \text{scal}_g > 0\}$$

If M has boundary, prescribe a metric h on ∂M and require product structure $\rightsquigarrow \mathcal{R}^+(M)_h$.

Uniqueness question

Assume $\mathcal{R}^+(M) \neq \emptyset$. What is the homotopy type of $\mathcal{R}^+(M)$?

Main tool: the index difference

From now on we assume all manifolds to be spin.

Index difference of Hitchin [Hit74]

Idea: define a map to K-theory using index theory and find elements of $\pi_k(\mathcal{R}^+(M)_h)$ that survive.
Result: For $g_0 \in \mathcal{R}^+(M)$

$$\mathrm{inddiff}_{g_0} : \mathcal{R}^+(M) \longrightarrow \Omega^{\infty+d+1}\mathbb{KO}$$

Idea:

- Index of the Dirac operator D_g will be zero for every $g \in \mathcal{R}^+(M)$ (Lichnerowicz).
- Compare two metrics instead: $tg_0 + (1-t)g_1$ yields a path of Fredholm operators.
- Start and end are invertible, since $g_0, g_1 \in \mathcal{R}^+(M)$
- The invertible operators make up a contractible space (Kuiper)

$\Rightarrow$ After taking the index, the path can be interpreted as a loop in K-theory

Let M^d be compact spin, $d \geqslant 6$, $h \in \mathcal{R}^+(\partial M)$, $g_0 \in \mathcal{R}^+(M)_h$ and $k \geqslant 0$.

Theorem (Botvinnik, Ebert, and Randal-Williams [BERW17])

The induced map

$$(\mathrm{inddiff}_{g_0})_* \colon \pi_k(\mathcal{R}^+(M)_h) \longrightarrow KO_{k+d+1=:m}(*) = \begin{cases} \mathbb{Z} & \text{if } m \equiv 0 \mod 4 \\ \mathbb{Z}/2 & \text{if } m \equiv 1,2 \mod 8 \\ 0 & \text{else} \end{cases}$$

is (rationally) surjective.

- Let $\mathrm{MTSpin}(d)$ be the Madsen–Tillmann spectrum
- There is a KO-orientation $\lambda_{-d}\colon \mathrm{MTSpin}(d) \to \Sigma^{-d}\mathbb{KO}$ ("**topological index**")

Theorem

There exists a map $\rho\colon \Omega^{\infty+1}\mathrm{MTSpin}(d) \to \mathcal{R}^+(M)_h$ *such that*

$$\Omega^{\infty+1}\mathrm{MTSpin}(d) \xrightarrow{\ \rho\ } \mathcal{R}^+(M^d)_h \xrightarrow{\ \mathrm{inddiff}_{g_0}\ } \Omega^{\infty+d+1}\mathbb{KO}$$

$$\Omega^{\infty+1}\mathrm{MTSpin}(d) \xrightarrow{\ \Omega^{\infty+1}\lambda_{-d}\ } \Omega^{\infty+d+1}\mathbb{KO}$$

is homotopy commutative.

(Rational) surjectivity of $\Omega^{\infty+1}\lambda_{-d} \implies$ Detection Theorem.

Improving the detection theorem

Improvement 1: Higher index theory

"richer" target for index difference $\Longrightarrow$ Potential to detect more classes of $\pi_k \mathcal{R}^+(M)_h$

Theorem (Ebert and Randal-Williams [ERW19a; ERW19b])

M^d spin, compact, $d \geqslant 6$, $G = \pi_1 M$. Then there exists ρ such that

$$\Omega^{\infty+1}(\mathrm{MTSpin}(d) \wedge BG_+) \xrightarrow{\rho} \mathcal{R}^+(M^d)_h \xrightarrow{\mathrm{inddiff}^G_{g_0}} \Omega^{\infty+d+1}\mathbb{KO}(C^*(G))$$

$$\Omega^{\infty+1}(\mathrm{MTSpin}(d) \wedge BG_+) \xrightarrow{\Omega^{\infty+1}\eta} \Omega^{\infty+d+1}\mathbb{KO}(C^*(G))$$

is homotopy commutative.

- $C^*(G)$ is the (reduced) group C^*-algebra.
- η is a generalisation of the KO-orientation using the Novikov assembly map.

Improvement 2: Extension to $d \geqslant 5$

Theorem (Perlmutter [Per17b])

The original detection and factorisation theorems also hold for $d \geqslant 5$.

Back to the roots:

- Extend the original Madsen and Weiss methods
- Use them to replace Galatius and Randal-Williams methods used by Botvinnik, Ebert, and Randal-Williams

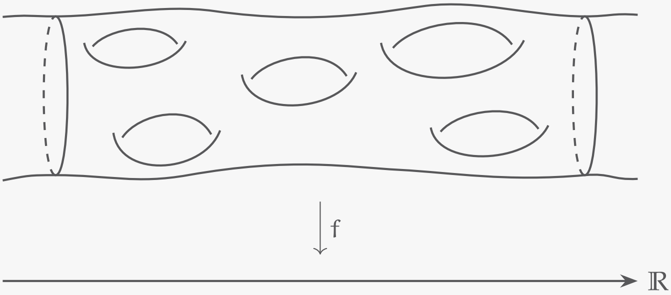
Theorem (B.)

Both improvements, i.e.

1. incorporation the fundamental group via higher index theory
2. extension to $d \geqslant 5$

can be carried out in unison.

Parametrised Morse Theory



Definition (Galatius and Randal-Williams)

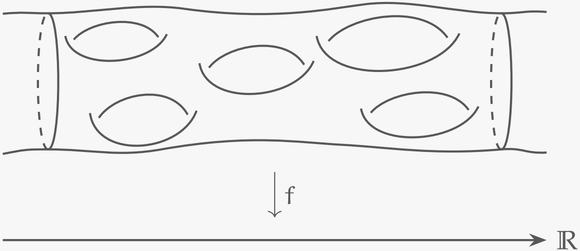
The **space of manifolds with one non-compact direction** is given by

$$\mathcal{D}_1 = \{(W, f) \mid f\colon W^d \to \mathbb{R} \text{ smooth and proper}\}$$

Theorem [GMTW09; GRW10]

$$\Omega^{\infty-1}MTO(d) \simeq \mathcal{D}_1 \simeq BCob$$

How can we turn a long d-manifold into a $(d-1)$-manifold?



Just take $f^{-1}(a)$ at a regular value $a \in \mathbb{R}$!

- Madsen and Weiss method: Non-destructive way to lower the dimension like this!
- Have to perform a “regularisation” to avoid critical points

Definition

Let $0 \leqslant k \leqslant \lfloor d/2 \rfloor$.
Let $\mathcal{D}^{[k]} \subset \mathcal{D}_1$ subspace with f Morse, Morse indices in $\{k, \dots, d-k\}$

The restriction on Morse indices was introduced by Perlmutter [Per17a].

Local model for critical points

Definition

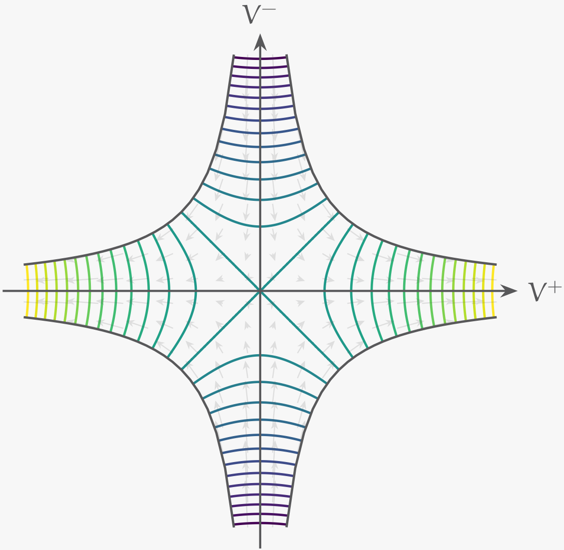
$V = V^+ \oplus V^-$ inner product space. The **saddle** is defined as

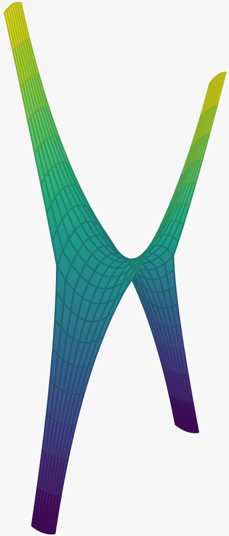
$$\mathrm{sdl}(V) := \left\{ v \in V \,\middle|\, \|v_+\|^2 \|v_-\|^2 \leqslant 1 \right\}$$

Canonical height function $f_V \colon \mathrm{sdl}(V) \to \mathbb{R}$ with unique critical point at the origin:

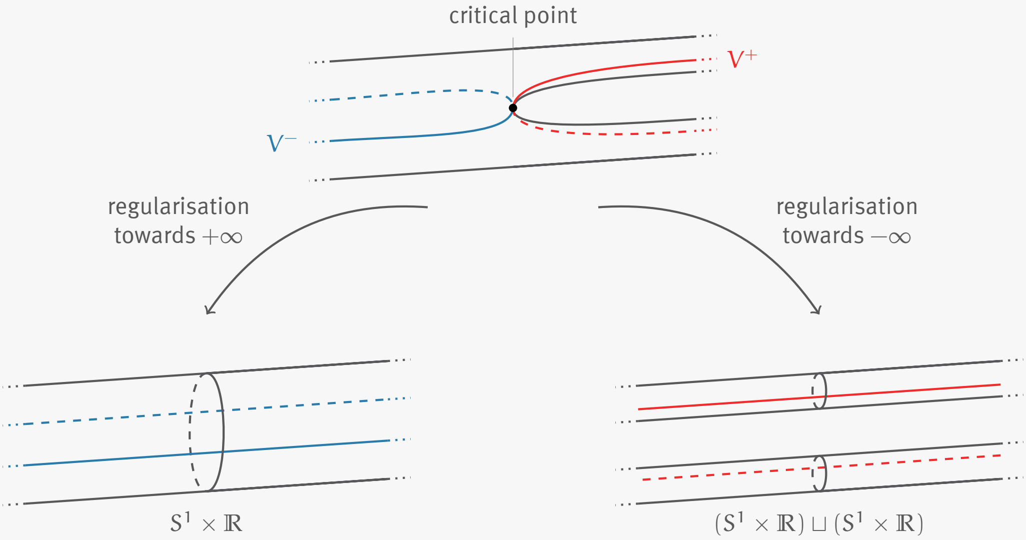
$$f_V(v) = \|v_+\|^2 - \|v_-\|^2$$

Plots of the saddle





Regularisation involves choices!



The homotopy colimit decomposition

For T a finite set:

Definition

Let $\mathcal{W}^{[k]}_T$ be the space of closed $(d-1)$-manifolds M equipped with surgery data indexed by T

- $\mathcal{K}^{[k]}$ custom indexing category
  - finite sets and injections over $\{k, \dots, d-k\}$
  - Morphisms know a sign for elements not in the image

Theorem [MW07; Per17a]

$$\mathcal{D}^{[k]} \simeq \operatorname*{hocolim}_{T \in \mathcal{K}^{[k]}} \mathcal{W}^{[k]}_{\theta,T}$$

- Add local data at critical points
- Encode regularisation choices using $\mathcal{K}^{[k]}$
- Perform the regularisation and take preimage at zero.

Definition (Morse Grassmannian)

Let $\mathrm{Gr}_{d,\theta}^{[k]}(\mathbb{R}^{d+N})$ denote the space of triples (V, l, σ) where

(i) $V \subset \mathbb{R}^{d+N}$ is an element of $\mathrm{Gr}_{d,\theta}(\mathbb{R}^{d+N})$

(ii) $l \colon V \to \mathbb{R}$ linear functional and $\sigma \colon V \times V \to \mathbb{R}$ symmetric bilinear form s.th.: If $l = 0$, then σ is non-degenerate with $k \leqslant \mathrm{index}(\sigma) \leqslant d - k$

As for the usual Grassmannian: Build a Thom spectrum

$$hW_{d,\theta}^{[k]} := \mathrm{Th}(-\gamma_\theta)$$

Theorem [MW07; Per17a]

The Pontryagin–Thom construction yields weak equivalences

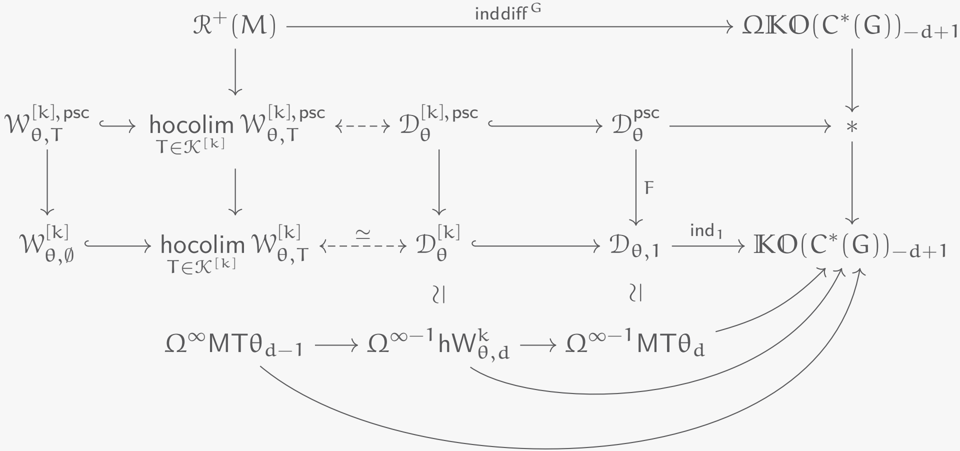
$$\Omega^{\infty-1} hW_{d,\theta}^{[k]} \simeq \mathcal{D}_\theta^{[k]} \simeq B\mathrm{Cob}_\theta^{\mathrm{mf},k}$$

- The proof is a very involved inductive argument in k with [MW07, Thm. 1.2] as base case.
- Homotopy colimit decomposition is central to the proof.

Proofsketch (back to psc)

Outline map of the proof

$\theta = BSpin(d) \times BG,\ k \geqslant 3 \Rightarrow d \geqslant 6$



Theorem

There is a fibration p *with fibre* $\mathcal{R}^+(M)$ *such that the diagram on the right is homotopy commutative and the induced map on homotopy fibres is* $\mathrm{inddiff}^G$.

$$\begin{array}{ccc} \mathcal{X} & \longrightarrow & * \\ \downarrow p & & \downarrow \\ \Omega^\infty MT\theta_{d-1} \longrightarrow \Omega^{\infty-1} hW^3_{\theta,d} & \xrightarrow{\Omega^{\infty-1}\eta_d} & \mathbb{KO}(C^*(\theta_d))_1 \end{array}$$

Taking the fibre transport of p at g_0 yields ρ and the following factorisation for $d-1 \geqslant 5$

$$\Omega^{\infty+1}MT\theta_{d-1} \longrightarrow \Omega^\infty hW^3_{\theta,d} \xrightarrow{\rho} \mathcal{R}^+(M)_h \xrightarrow{\mathrm{inddiff}^G_{g_0}} \Omega^{\infty+d}\mathbb{KO}(C^*(G))$$

$$\Omega^{\infty+1}MT\theta_{d-1} \xrightarrow{\Omega^{\infty+1}\eta_{d-1}} \Omega^{\infty+d}\mathbb{KO}(C^*(G))$$

Remark: Similar diagrams are used in [BERW17; ERW19a; ERW19b]

Bonus: The action $\mathrm{Diff}(M) \curvearrowright \mathcal{R}^+(M)$

$$\begin{array}{ccccc}
\mathcal{R}^+(M) & \xlongequal{\quad\quad} & \mathcal{R}^+(M) & \xlongequal{\quad\quad} & \mathcal{R}^+(M) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{R}^+(M)/\!\!/\mathrm{Diff}^\theta(M) & \longrightarrow & \mathcal{W}^{[3],\mathrm{psc}}_{\theta,\emptyset} & \hookrightarrow & \operatorname*{hocolim}_{T\in\mathcal{K}^{[3]}} \mathcal{W}^{[3],\mathrm{psc}}_{\theta,T} \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{BDiff}^\theta(M) & \hookrightarrow & \mathcal{W}^{[3]}_{\theta,\emptyset} & \hookrightarrow & \operatorname*{hocolim}_{T\in\mathcal{K}^{[3]}} \mathcal{W}^{[3]}_{\theta,T} \simeq \Omega^{\infty-1} hW^3_{\theta,d}
\end{array}$$

Theorem (B.)

$M^{d-1} \in \mathcal{W}^{[3],\mathrm{psc}}_{\theta,\emptyset}$ simply connected (i.e. $\theta = \mathrm{BSpin}$) and $g_0 \in \mathcal{R}^+(M)$. The orbit map $\sigma_{g_0}\colon \mathrm{Diff}^{\mathrm{Spin}}(M) \to \mathcal{R}^+(M)$ factors in π_k through a map

$$\pi_k(\mathrm{Diff}^{\mathrm{Spin}}(M)) \simeq \pi_{k+1}(\mathrm{BDiff}^{\mathrm{Spin}}(M)) \longrightarrow \pi_{k+1}(MT\mathrm{Spin}(d-1))$$

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