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Atiyah's KR-theory and real Bott periodicity

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Motivation

- Topological K-theory fulfils **Bott periodicity**

$$\mathrm{KU}(X) \cong \mathrm{KU}^{-2}(X) \qquad \mathrm{KO}(X) \cong \mathrm{KO}^{-8}(X)$$

- While there are quite a few proofs for the complex case in the literature (Hatcher [Hat17], Atiyah [Ati67], Aguilar, Gitler, and Prieto [AGP08], Switzer [Swi02]) ...
- ...the real case is often neglected:[1]

2.4. Bott Periodicity in the Real Case [not yet written]

[1] at least in textbooks

Atiyah's KR-theory

Introduced in "K-theory and reality," Atiyah [Ati66]

1. KR-theory is some kind of mixture of complex (KU) and real K-theory (KO).
2. KR has a $(1,1)$-periodicity analogous to the 2-periodicity of KU ...
3. ...out of which the 2-periodicity and 8-periodicity of KU resp. KO can be derived!
4. Using KR-theory one can prove that the Atiyah-Bott-Shapiro map $\alpha\colon A_* \to \mathrm{KO}^*(\mathrm{pt})$ is an isomorphism.
5. When doing index theory KR turns out to be the "right" framework for the index theorem of real operators.

We will cover all except the last point in this talk, focusing on items 3. and 4.

K-THEORY AND REALITY

By M. F. ATIYAH

[Received 9 August 1966]

Introduction

The K-theory of complex vector bundles (**2**, **5**) has many variants and refinements. Thus there are:

(1) K-theory of real vector bundles, denoted by KO,
(2) K-theory of self-conjugate bundles, denoted by KC (**1**) or KSC (**7**),
(3) K-theory of G-vector bundles over G-spaces (**6**), denoted by K_G.

In this paper we introduce a new K-theory denoted by KR which is, in a sense, a mixture of these three. Our definition is motivated partly by analogy with real algebraic geometry and partly by the theory of real elliptic operators. In fact, for a thorough treatment of the index problem for real elliptic operators, our KR-theory is essential. On the other hand, from the purely topological point of view, KR-theory has a number of advantages and there is a strong case for regarding it as the primary theory and obtaining all the others from it. One of the main purposes of this paper is in fact to show how KR-theory leads to an elegant proof of the periodicity theorem for KO-theory, starting essentially from the periodicity theorem for K-theory as proved in (**3**). On the way we also encounter, in a natural manner, the self-conjugate theory and various exact sequences between the different theories. There is here a considerable overlap with the thesis of Anderson (**1**) but, from our new vantage point, the relationship between the various theories is much easier to see.

Recently Karoubi (**8**) has developed an abstract K-theory for suitable categories with involution. Our theory is included in this abstraction but its particular properties are not developed in (**8**), nor is it exploited to simplify the KO-periodicity.

The definition and elementary properties of KR are given in § 1. The periodicity theorem and general cohomology properties for KR are discussed in § 2. Then in § 3 we introduce various derived theories—KR with coefficients in certain spaces—ending up with the periodicity theorem for KO. In § 4 we discuss briefly the relation of KR with Clifford algebras on the lines of (**4**), and in particular we establish a lemma which is used in § 3. The significance of KR-theory for the topological study of real elliptic operators is then briefly discussed in § 5.

Quart. J. Math. Oxford (2), 17 (1966), 367–86.

Basics of KR-theory

The *Real* category

Definition 1

A **Real space** is a topological space X with a continuous involution $\tau\colon X \to X$. A **Real map** commutes with the involutions.

- We often think of τ as complex conjugation and hence use the bar notation $x \mapsto \overline{x}$ in general.
- Subspace of Real points:

$$X_{\mathbb{R}} := \{x \in X \mid \overline{x} = x\}$$

- A Real subspace is closed under $\tau \Rightarrow$ basepoints have to be fixed points!

Examples:

(i) Any X with $\tau = \mathrm{id}$

(ii) $\mathbb{C}$ with $\tau(x) = \overline{x}$

(iii) More generally: $\mathbb{R}^{p,q} := \mathbb{R}^q \oplus i \cdot \mathbb{R}^p$ with $\tau(x, y) = (x, -y)$. In particular $\mathbb{C}^k \cong \mathbb{R}^{k,k}$ as Real spaces.

(iv) Corresponding spheres and balls: $S^{p,q} \subset B^{p,q} \subset \mathbb{R}^{p,q}$

But wait: That's just the definition of a G-space for $G = \mathbb{Z}_2$! – Right, but:

Definition 2

A **Real vector bundle** over a Real space X is a complex vector bundle E over X such that

(i) E is a Real space with involution τ,

(ii) the projection $E \to X$ commutes with the involutions,

(iii) the map $\tau\colon E_x \to E_{\overline{x}}$ is anti-linear for every $x \in X$

It is clear where to go from here: Prove everything you know about complex bundles for Real bundles (direct sums, tensor products, etc.) Then define for X compact and Real

$$KR(X) := \mathrm{Groth}\big(\mathrm{Vect}^R(X)\big)$$

where $\mathrm{Vect}^R(X)$ is the monoid of isomorphism classes of Real vector bundles with the operation $\oplus$.

Connection with KO

From now on: (X, τ) compact Real space. Let $\mathscr{F}(X)$ be the category of Real vector bundles over X, $\mathscr{E}(X)$ the one of real vector bundles.

Theorem 3

If X has trivial involution, then there is an equivalence of categories $\mathscr{E}(X) \simeq \mathscr{F}(X)$

Proof: Consider

$$\begin{array}{rcl} \mathscr{E}(X) & \xrightarrow{\simeq} & \mathscr{F}(X) \\ E & \longmapsto & E \otimes_{\mathbb{R}} \mathbb{C} \\ F_{\mathbb{R}} & \longleftarrow\!\!| & F \end{array}$$

□

Corollary 4

If X has trivial involution, then $KR(X) \simeq KO(X)$.

Connection with KU

$S^{1,0} = \{\pm 1\}$ with the involution interchanging the two points.

Theorem 5

There is a natural isomorphism $\mathrm{KR}(X \times S^{1,0}) \cong \mathrm{KU}(X)$, *no matter the choice of involution* τ *on* X*!*

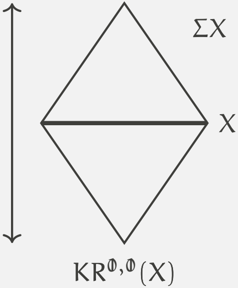
PROOF: Restriction to $X \times \{+1\}$ gives a *complex* bundle $E \to X$. Put $\tau^* \overline{E}$ over the other copy of X to get back the Real bundle. □

Proceed to define reduced and relative KR-groups exactly as in the classical case.

Suspension Groups

Recall: $K^{-1}(X) = K(\Sigma X)$

- For KR there are **two** choices of involution on the suspension!
- We may use $t \mapsto t$ on the real axis $\rightsquigarrow KR^{0,1}(X)$.
- Or $t \mapsto -t$ on the real axis $\rightsquigarrow KR^{1,0}(X)$
- In general: $KR^{p,q}$



Definition 6

For $p, q \geqslant 0$ we define

$$KR^{p,q}(X, Y) \coloneqq KR((X \setminus Y) \times \mathbb{R}^{p,q})$$

- The "usual suspension groups" are recovered by $KR^{-q} \coloneqq KR^{0,q}$.
- $KR^{0,0}(X, Y) = KR(X, Y)$

Products and $(1,1)$-periodicity

Via projections and tensor products one defines **external products**:

$$\mathrm{KR}^{p,q}(X,Y) \otimes \mathrm{KR}^{s,t}(X',Y') \to \mathrm{KR}^{p+s,q+t}(X \times X', X \times Y' \cup Y \times X')$$

Let H be the dual of the tautological bundle over $\mathbb{C}P^1$. H is a Real bundle.

Theorem 7 ($(1,1)$-periodicity)

The element

$$b = [H] - \mathbb{1} \in \mathrm{KR}^{1,1}(\mathrm{pt}) = \mathrm{KR}(B^{1,1}, S^{1,1}) = \widetilde{\mathrm{KR}}(\mathbb{C}P^1) = \mathrm{KR}(\mathbb{R}^{1,1})$$

*is called the **Bott element** (or **reduced Hopf bundle**). Multiplication by b gives an isomorphism*

$$\beta\colon \mathrm{KR}^{p,q}(X,Y) \xrightarrow{\cong} \mathrm{KR}^{p+1,q+1}(X,Y)$$

PROOF: Modify the "elementary proof" of ATIYAH and BOTT [AB64] for the Real case. See [Ati66]. □

Set $\mathrm{KR}^p := \mathrm{KR}^{p,0}$ for $p \geqslant 0$. Then $\mathrm{KR}^{p,q} \cong \mathrm{KR}^{p-q}$

Applications of KR-theory

KR-theory with coefficients

Let Y be some compact Real space. The functor $X \mapsto KR(X \times Y)$ defines a new theory called **KR-theory with coefficients in Y.**

- $KR(X \times S^{1,0}) \cong KU(X)$ suggests that $Y = S^{p,0}$ might be interesting.
- If F is a functor as above we say, that F has period q, if $F \cong F^{-q}$, where F^{-q} is defined via suspensions with trivial involution.

Lemma 8

KR-theory with coefficients in $S^{p,0}$ has the following period as indicated in the table.

p	period
1	2
2	4
4	8

Intuitively:

$$p = 1 \rightsquigarrow KU$$
$$p = 2 \rightsquigarrow KSC \quad \text{(K-theory of self conj. bundles)}$$
$$p = 4 \rightsquigarrow KO$$

Proof: For $p = 1, 2, 4$ we view $\mathbb{R}^p$ as $\mathbb{R}, \mathbb{C}, \mathbb{H}$. Now consider

$$\begin{aligned} \mu_p : X \times S^{p,0} \times \mathbb{R}^{0,p} &\longrightarrow X \times S^{p,0} \times \mathbb{R}^{p,0} \\ (x, s, u) &\longmapsto (x, s, us) \end{aligned}$$

μ_p is Real. Replacing X by a suitable suspension we get

$$\mu_p^* : KR^{p,q}(X \times S^{p,0}) \xrightarrow{\cong} KR^{0,p+q}(X \times S^{p,0})$$

Applying the $(1,1)$-periodicity isomorphism β p times in the case $p = q$ we arrive at

$$\gamma_p \coloneqq \mu_p^* \circ \beta^p : KR(X \times S^{p,0}) \xrightarrow{\cong} KR^{0,2p}(X \times S^{p,0}) = KR^{-2p}(X \times S^{p,0}) \qquad \square$$

γ_p is $KR(X \times S^{p,0})$-linear and therefore given by multiplication with

$$c_p = \gamma_p(\mathbb{1}) = \mu_p^*(\mathbb{1} * b^p) \in KR^{-2p}(S^{p,0})$$

Corollary 9 (Complex Bott periodicity)

We have $KU(X) \cong KU^{-2}(X)$.

PROOF: Obivous, since $KR(X \times S^{1,0}) \cong KU(X)$. □

Consider the inclusion $\iota\colon \mathbb{R} = \mathbb{R}^{0,1} \hookrightarrow \mathbb{R}^{1,1} = \mathbb{C}$.

$$\eta = \iota^*(b) = \iota^*([H] - \mathbb{1}) \in KR^{0,1}(pt)$$

is called the **reduced real Hopf bundle** ($\iota^*(H)$ is the real Hopf bundle).

Theorem 10 (Gysin Sequence)

There is an exact sequence

$$\dots \longrightarrow KR^{p-q}(X) \xrightarrow[\cdot(-\eta)^p]{\chi} KR^{-q}(X) \xrightarrow{\pi^*} KR^{-q}(X \times S^{p,0}) \xrightarrow{\delta} \dots$$

where $\pi\colon S^{p,0} \to pt$ *is the projection and* χ *is the product with* $(-\eta)^p$.

The KO-KU-sequence

Gysin sequence for $p = 1$:

$$\dots \longrightarrow KR^{1-q}(X) \xrightarrow{\chi} KR^{-q}(X) \xrightarrow{\pi^*} KU^{-q}(X) \xrightarrow{\delta} KR^{2-q}(X) \longrightarrow \dots$$

With trivial involution on X:

$$\dots \longrightarrow KO^{1-q}(X) \xrightarrow{\chi} KO^{-q}(X) \xrightarrow{c} KU^{-q}(X) \xrightarrow{\delta} KO^{2-q}(X) \longrightarrow \dots$$

where c is the map induced by complexification. This is the **KO-KU-sequence**.

Clifford modules

We denote with $C\ell_p$ the real Clifford algebra with generators $e_1, \dots, e_p$ such that $e_i^2 = -1$ and $e_i e_j = -e_j e_i$ for $i \neq j$.

Goal

Define a "suitable" group of $C\ell_p$-modules.

- We denote with $\widehat{\mathfrak{M}}_p$ the free abelian group generated by the irreducible $\mathbb{Z}_2$-graded $C\ell_p$-modules.

The inclusion $i\colon C\ell_p \hookrightarrow C\ell_{p+1}$ induces a **restriction of scalars** homomorphism

$$i^*\colon \widehat{\mathfrak{M}}_{p+1} \longrightarrow \widehat{\mathfrak{M}}_p$$

and we define $A_p := \operatorname{coker} i^*$.

- There are complex analogues over the complex Clifford algebras $\mathbb{C}\ell_p$, denoted by $\widehat{\mathfrak{M}}^c$ and A_p^c.
- Via the graded tensor product we get graded rings $\widehat{\mathfrak{M}}_*$ and A_*.

p	$C\ell_p$	$\widehat{\mathfrak{M}}_p$	A_p	$\widehat{\mathfrak{M}}^c_p$	A^c_p
1	$\mathbb{C}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}$	0
2	$\mathbb{H}$	$\mathbb{Z}$	$\mathbb{Z}_2$	$\mathbb{Z}\oplus\mathbb{Z}$	$\mathbb{Z}$
3	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{Z}$	0	$\mathbb{Z}$	0
4	$M_2(\mathbb{H})$	$\mathbb{Z}\oplus\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}$	$\mathbb{Z}$
5	$M_4(\mathbb{C})$	$\mathbb{Z}$	0	$\mathbb{Z}$	0
6	$M_8(\mathbb{R})$	$\mathbb{Z}$	0	$\mathbb{Z}\oplus\mathbb{Z}$	$\mathbb{Z}$
7	$M_8(\mathbb{R})\oplus M_8(\mathbb{R})$	$\mathbb{Z}$	0	$\mathbb{Z}$	0
8	$M_{16}(\mathbb{R})$	$\mathbb{Z}\oplus\mathbb{Z}$	$\mathbb{Z}$	$\mathbb{Z}\oplus\mathbb{Z}$	$\mathbb{Z}$

Theorem 11 (Atiyah, Bott, and Shapiro [ABS64])

A_* is the anticommutative graded ring generated by a unit $\mathbb{1}\in A_0$ and elements $\xi\in A_1$, $\mu\in A_4$, $\lambda\in A_8$ with the relations $2\xi=0,\quad \xi^3=0,\quad \mu^2=4\lambda$
Complex case: $A^c_*\cong\mathbb{Z}[\mu^c]$ *with* $\mu^c\in A^c_2$.

The Atiyah-Bott-Shapiro map

Consider

$$\begin{aligned} \widehat{\mathfrak{M}}_p &\longrightarrow KO(B^p, S^p) = KO^{-p}(\mathrm{pt}) \\ M = M^0 \oplus M^1 &\longmapsto [M^0, M^1; \sigma] \end{aligned}$$

where σ is defined by Clifford multiplication with $x \in B^p \subset \mathbb{R}^p \subset C\ell_p^1$, which gives a map $M^0 \to M^1$, that is an isomorphism for $x \in S^p$. These maps induce a homomorphism of graded rings

$$\alpha\colon A_* \longrightarrow KO^*(\mathrm{pt})$$

called the **Atiyah-Bott-Shapiro map**.

- The same construction can be carried out in the complex case yielding $\alpha^c\colon A_*^c \to KU^*(\mathrm{pt})$.
- Furthermore there is a Real variant with $C\ell_{p,q}$ in place of $C\ell_p$ (which is of little avail for us)

The ABS map is an isomorphism – Preliminaries

The original proof of Atiyah, Bott, and Shapiro [ABS64] is rather unsatisfactory, since it requires complete knowledge of the ring $\mathrm{KO}^*(\mathrm{pt})$.

- Our main tool: The KO-KU-sequence and knowledge of A_*. Furthermore we need:
- The complex ABS map is an isomorphism: $\alpha^c \colon A_*^c \xrightarrow{\cong} \mathrm{KU}^{-*}(\mathrm{pt})$
- Complexification on the level of Clifford modules takes the following form

$$\begin{array}{ccc} A_4 & \longrightarrow & A_4^c \\ \mu & \longmapsto & 2(\mu^c)^2 \end{array} \qquad \begin{array}{ccc} A_8 & \longrightarrow & A_8^c \\ \lambda & \longmapsto & (\mu^c)^4 \end{array}$$

- Some "activation energy" to get the KO-KU-sequence going: $\alpha\colon A_1 \xrightarrow{\cong} \mathrm{KO}^{-1}(\mathrm{pt}) \cong \mathbb{Z}_2$ (use the definition of α to show that $\alpha(\xi) = \eta$, then argue, that η is the nontrivial element)

The ABS map is an isomorphism – Proof

Consider the following portion of the KO-KU-sequence:

$$
\begin{array}{ccccccccccccc}
KU^{-3} & \longrightarrow & KO^{-1} & \xrightarrow{\cdot\eta} & KO^{-2} & \xrightarrow[0]{c} & KU^{-2} & \overset{\delta}{\underset{\cdot(\pm 2)}{\hookrightarrow}} & KO^{0} & \overset{\cdot\eta}{\twoheadrightarrow} & KO^{-1} & \longrightarrow & KU^{-1} \\
\| & & \wr\| & & \wr\| & & \wr\| & & \wr\| & & \wr\| & & \| \\
0 & & \mathbb{Z}_2 & & \mathbb{Z}_2\{\eta^2\} & & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z}_2 & & 0
\end{array}
$$

The next bit of the sequence has the following form:

$$
\begin{array}{ccccccccccccc}
 & & & & A_4 & \overset{c}{\underset{\cdot 2}{\hookrightarrow}} & A_4^c & & & & & & \\
 & & & & \downarrow \alpha & & \cong \downarrow \alpha^c & & & & & & \\
KU^{-5} & \longrightarrow & KO^{-3} & \xrightarrow{\cdot\eta} & KO^{-4} & \overset{c}{\underset{\cdot 2}{\hookrightarrow}} & KU^{-4} & \overset{\delta}{\twoheadrightarrow} & KO^{-2} & \overset{\cdot\eta}{\twoheadrightarrow} & KO^{-3} & \longrightarrow & KU^{-3} \\
\| & & \| & & \wr\| & & \wr\| & & \wr\| & & \| & & \| \\
0 & & 0 & & \mathbb{Z} & & \mathbb{Z} & & \mathbb{Z}_2\{\eta^2\} & & 0 & & 0
\end{array}
$$

$\eta^3 = 0$, since $\xi^3 = 0$ and $\alpha(\xi) = \eta$.

$\Rightarrow$ α is an isomorphism in degrees up to 4. Similar arguments give the remaining cases $5, 6, 7, 8$. $\square$

Proof of real Bott periodicity – Preliminaries

As $\eta^3 = 0$ the Gysin sequence for $p \geqslant 3$ splits up:

$$0 \longrightarrow KR^{-q}(X) \xrightarrow{\pi^*} KR^{-q}(X \times S^{p,0}) \xrightarrow{\delta} KR^{p+1-q}(X) \longrightarrow 0$$

Recall: Multiplication by $c_4 \in KR^{-8}(S^{4,0})$ gave the 8-periodicity of $KR(X \times S^{4,0})$.

Lemma 12

Let $\mathbb{1}$ *denote the identity of* $KR(S^{4,0})$. *Then with* λ *the generator of* A_8 *and* $\mathrm{pr}\colon S^{4,0} \to \mathrm{pt}$

$$c_4 = \alpha(\lambda) * \mathbb{1} = \mathrm{pr}^*(\alpha(\lambda)) \in KR^{-8}(S^{4,0})$$

Proof:

$$0 \longrightarrow KO^{-8}(\mathrm{pt}) \xrightarrow{\mathrm{pr}^*} KR^{-8}(S^{4,0}) \longrightarrow KO^{-3}(\mathrm{pt}) \longrightarrow 0$$

$KO^{-3}(\mathrm{pt}) = 0$ by the ABS *Iso*morphism, so $c_4 = \mathrm{pr}^*(n \cdot \alpha(\lambda))$. But $n = \pm 1$, since otherwise multiplication by c_4 could not induce isomorphisms for general X. □

Proof of real Bott periodicity – Final Step

Theorem 13

Let $\lambda \in A_8$, $\alpha(\lambda) \in KO^{-8}(pt) = KR^{-8}(pt)$ *as above. Then multiplication by* $\alpha(\lambda)$ *induces an iso*

$$KR(X) \xrightarrow{\cong} KR^{-8}(X)$$

Proof: Consider the diagram with all vertical maps given by multiplication with $\alpha(\lambda)$

$$\begin{array}{ccccccccc}
0 & \longrightarrow & KR^{-q}(X) & \longrightarrow & KR^{-q}(X \times S^{4,0}) & \longrightarrow & KR^{5-q}(X) & \longrightarrow & 0 \\
 & & \downarrow \phi_{-q} & & \cong \downarrow \psi_{-q} & & \downarrow \phi_{5-q} & & \\
0 & \longrightarrow & KR^{-q-8}(X) & \longrightarrow & KR^{-q-8}(X \times S^{4,0}) & \longrightarrow & KR^{-3-q}(X) & \longrightarrow & 0
\end{array}$$

By Lemma 12 the map ψ_{-q} is an isomorphism for every q. Hence ϕ_{-q} is injective for every q, in particular: ϕ_{5-q} is injective $\Rightarrow \phi_{-q}$ surjective. □

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