Spaces of PSC metrics and parametrised Morse theory

A detection theorem for $d \geqslant 5$ and higher index theory

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Recap on positive scalar curvature metrics

Let (M, g) be a Riemannian manifold.

- Central notion in differential geometry: Riemannian curvature tensor associated to g.
- Tensor contraction turns this into **scalar curvature** $\rightsquigarrow$ smooth function $\mathrm{scal}_g : M \to \mathbb{R}$

Existence question

Given a smooth manifold M. Does M admit a metric with $\mathrm{scal}_g > 0$?

- There are many (topological) obstructions to admitting psc ($\hat{A}$-genus, α-invariant, enlargeability, ...).
- Index theory provides a bridge to topology, but only works in the presence of spin structures.

The space of psc metrics

Definition (Space of metrics)

Let $\mathcal{R}(M)$ be the **space of Riemannian metrics** (with the C^∞-topology).

Note: $\mathcal{R}(M)$ is convex, so $\mathcal{R}(M) \simeq *$

Definition (Space of psc metrics)

$$\mathcal{R}^+(M) \coloneqq \left\{ g \in \mathcal{R}(M) \;\middle|\; \mathrm{scal}_g > 0 \right\}$$

If M has boundary, prescribe a metric h on ∂M and require product structure $\rightsquigarrow \mathcal{R}^+(M)_h$.

Uniqueness question

Assume $\mathcal{R}^+(M) \neq \emptyset$. What is the homotopy type of $\mathcal{R}^+(M)$?

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First main tool: Gromov–Lawson–Chernysh surgery

Let $\phi\colon V^k \times \mathbb{R}^{d-k} \hookrightarrow M^d$ be an open embedding and $h_V \in \mathcal{R}^+(V)$

Definition

$$\mathcal{R}^+(M, \phi) \coloneqq \text{ space of psc metrics with prescribed metric on } \operatorname{im}(\phi)$$

Theorem (Chernysh [Che04])

If $d - k \geqslant 3$, then the inclusion

$$\mathcal{R}^+(M, \phi) \longrightarrow \mathcal{R}^+(M)$$

is a weak equivalence. Similarly for $\mathcal{R}^+(M)_h$

Corollary (Surgery equivalence)

For $V = S^k$ and $d - k, k \geqslant 3$ there is a preferred class of weak homotopy equivalences

$$\mathcal{R}^+(M) \simeq \mathcal{R}^+(M_\phi)$$

where M_ϕ denotes the surgered manifold.

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Second main tool: the index difference

From now on we assume all manifolds to be spin.

Index difference of Hitchin [Hit74]

Idea: define a map to K-theory using index theory and find elements of $\pi_k(\mathcal{R}^+(M)_h)$ that survive.
Result: For $g_0 \in \mathcal{R}^+(M)$

$$\mathrm{inddiff}_{g_0} : \mathcal{R}^+(M) \longrightarrow \Omega^{\infty+d+1}\mathbb{KO}$$

- Index of the Dirac operator D_g will be zero for every $g \in \mathcal{R}^+(M)$ (Lichnerowicz).
- Compare two metrics instead: $tg_0 + (1-t)g_1$ yields a path of Fredholm operators.
- Start and end are invertible, since $g_0, g_1 \in \mathcal{R}^+(M)$
- The invertible operators make up a contractible space (Kuiper)

⇒ After taking the index, the path can be interpreted as a loop in K-theory

(in a proper implementation of this idea I would use KK-cycles)

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The detection theorem

Let M^d be compact spin, $d \geqslant 6$, $h \in \mathcal{R}^+(\partial M)$, $g_0 \in \mathcal{R}^+(M)_h$ and $k \geqslant 0$.

Theorem (Botvinnik, Ebert, and Randal-Williams [BERW17])

The induced map

$$(\mathrm{inddiff}_{g_0})_* \colon \pi_k\big(\mathcal{R}^+(M)_h\big) \longrightarrow KO_{k+d+1=:m}(*) = \begin{cases} \mathbb{Z} & \text{if } m \equiv 0 \mod 4 \\ \mathbb{Z}/2 & \text{if } m \equiv 1,2 \mod 8 \\ 0 & \text{else} \end{cases}$$

is (rationally) surjective.

The factorisation theorem

- Let $\mathrm{MTSpin}(d)$ be the Madsen–Tillmann spectrum
- There is a KO-orientation $\lambda_{-d}\colon \mathrm{MTSpin}(d) \to \Sigma^{-d}\mathbb{KO}$ (“**topological index**”)
- $\widehat{\mathcal{A}} = \Omega^\infty(\lambda_{-d})$

Theorem

There exists a map $\rho\colon \Omega^{\infty+1}\mathrm{MTSpin}(d) \to \mathcal{R}^+(M)_h$ *such that*

$$\begin{tikzcd}
\Omega^{\infty+1}\mathrm{MTSpin}(d) \arrow[r, "\rho"] \arrow[rr, bend right, "\Omega\widehat{\mathcal{A}}"'] & \mathcal{R}^+(M^d)_h \arrow[r, "\mathrm{inddiff}_{g_0}"] & \Omega^{\infty+d+1}\mathbb{KO}
\end{tikzcd}$$

is homotopy commutative.

(Rational) surjectivity of $\widehat{\mathcal{A}}_* \implies$ Detection Theorem.

Improving the detection theorem

Improvement 1: Higher index theory

“richer” target for index difference $\Longrightarrow$ Potential to detect more classes of $\pi_k \mathcal{R}^+(M)_h$

Higher index difference

$$\mathrm{inddiff}^{G}_{g_0} \colon \mathcal{R}^+(M)_h \longrightarrow \Omega^{\infty+d+1}\mathbb{KO}(C^*(G))$$

where $C^*(G)$ is the (reduced) group C^*-algebra for $G = \pi_1 M$.

- The spin Dirac operator on M can be twisted by a bundle E, which introduces an extra term in the Lichnerowicz formula (see enlargeability and Llarull’s theorem).
- Rosenberg: Twist with the flat **Miščenko–Fomenko line bundle** $\mathcal{L}_G \coloneqq EG \times_G C^*(G) \to BG$
- twisted Dirac operator $D_{\mathcal{L}_G}$ acts on a Hilbert-$C^*(G)$-module and has an index in $KO_d(C^*(G))$

Analogue for the topological index, using the Novikov assembly map ν:

$$\eta \colon \mathrm{MTSpin}(d) \wedge BG_+ \xrightarrow{\lambda_{-d} \wedge \mathrm{id}} \Sigma^{-d}\mathbb{KO} \wedge BG_+ \xrightarrow{\Sigma^{-d}\nu} \Sigma^{-d}\mathbb{KO}(C^*(G))$$

The Detection theorem for higher index theory

Theorem (Ebert and Randal-Williams [ERW19a; ERW19b])

M^d spin, compact, $d \geqslant 6$. Then there exists ρ such that

$$\begin{tikzcd}
\Omega^{\infty+1}(MTSpin(d) \wedge BG_+) \arrow[r, "\rho"] \arrow[rr, bend right, "\Omega^{\infty+1}\eta"'] & \mathcal{R}^+(M^d)_h \arrow[r, "\mathrm{inddiff}^G_{g_0}"] & \Omega^{\infty+d+1}\mathbb{K}\mathbb{O}(C^*(G))
\end{tikzcd}$$

is homotopy commutative

To derive detection results: study the assembly map ⇝ assumptions on G.

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Improvement 2: Extension to $d \geqslant 5$

Theorem (PERLMUTTER [Per17b])

The original detection and factorisation theorems also hold for $d \geqslant 5$.

- Back to the roots: Replace GRW methods by original methods of MADSEN and WEISS [MW07]
- Series of two preprints (sadly, he left mathematics):
    1. Extension of MADSEN and WEISS [MW07] methods
       (to reprove high-dimensional MW theorem of GALATIUS and RANDAL-WILLIAMS [GRW14])
    2. Application to PSC

Theorem (B.)

Both improvements, i.e.

1. *incorporation the fundamental group via higher index theory*
2. *extension to* $d \geqslant 5$

can be carried out in unison.

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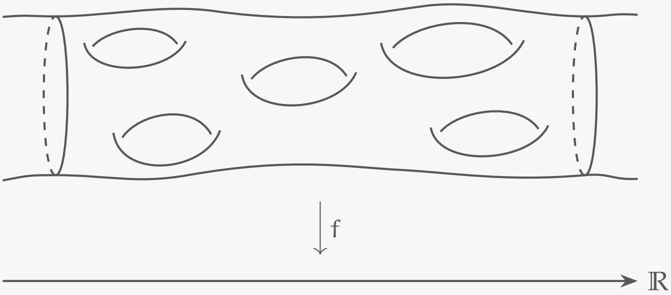
Parametrised Morse Theory

Spaces of manifolds: Long manifolds



Definition (Galatius and Randal-Williams)

The **space of manifolds with one non-compact direction** is given by

$$\mathcal{D}_1 = \{(W, f) \mid f\colon W^d \to \mathbb{R} \text{ smooth and proper}\}$$

Theorem [GMTW09; GRW10]

$$\Omega^{\infty-1}\mathrm{MTO}(d) \simeq \mathcal{D}_1 \simeq \mathrm{BCob}$$

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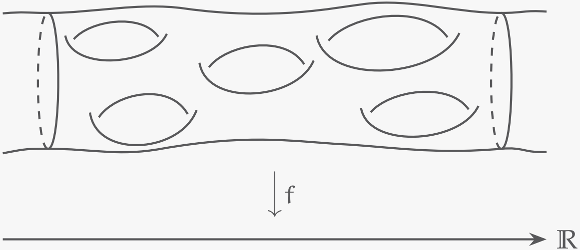
How can we turn a long d-manifold into a $(d-1)$-manifold?



Just take $f^{-1}(a)$ at a regular value $a \in \mathbb{R}$!

- Madsen and Weiss method: Non-destructive way to lower the dimension like this!
- Have to perform a “regularisation” to avoid critical points

Definition

Let $0 \leqslant k \leqslant \lfloor d/2 \rfloor$.
Let $\mathcal{D}^{[k]} \subset \mathcal{D}_1$ subspace with f Morse, Morse indices in $\{k, \ldots, d-k\}$ and $\ell \colon W \to BO(d)$ $(k-1)$-connected.

The restriction on Morse indices was introduced by Perlmutter [Per17a].

Local model for critical points

Definition

$V = V^+ \oplus V^-$ inner product space. The **saddle** is defined as

$$\mathrm{sdl}(V) := \left\{ v \in V \,\middle|\, \|v_+\|^2 \|v_-\|^2 \leqslant 1 \right\}$$

Canonical height function on $\mathrm{sdl}(V)$ with unique critical point at the origin:

$$f_V(v) = \|v_+\|^2 - \|v_-\|^2$$

Regularisation: Remove V^+ or V^- and adjust height function such that height approaches $+\infty$ near V^+.

Definition

$\mathcal{L}^{[k]}$ has the same data as $\mathcal{D}^{[k]}$ with embedded saddles around all critical points, such that the height functions f_V and f are compatible.

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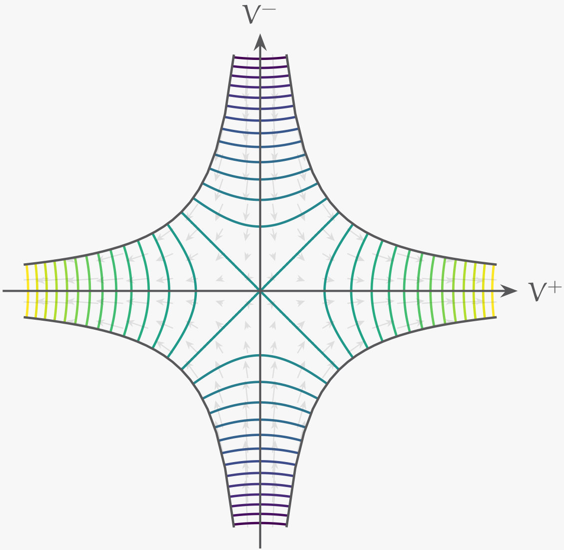
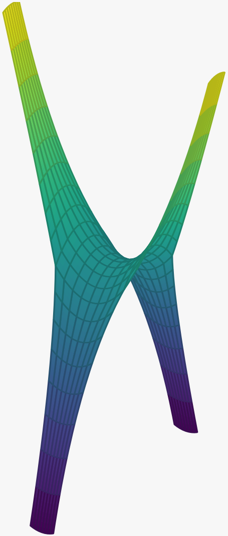
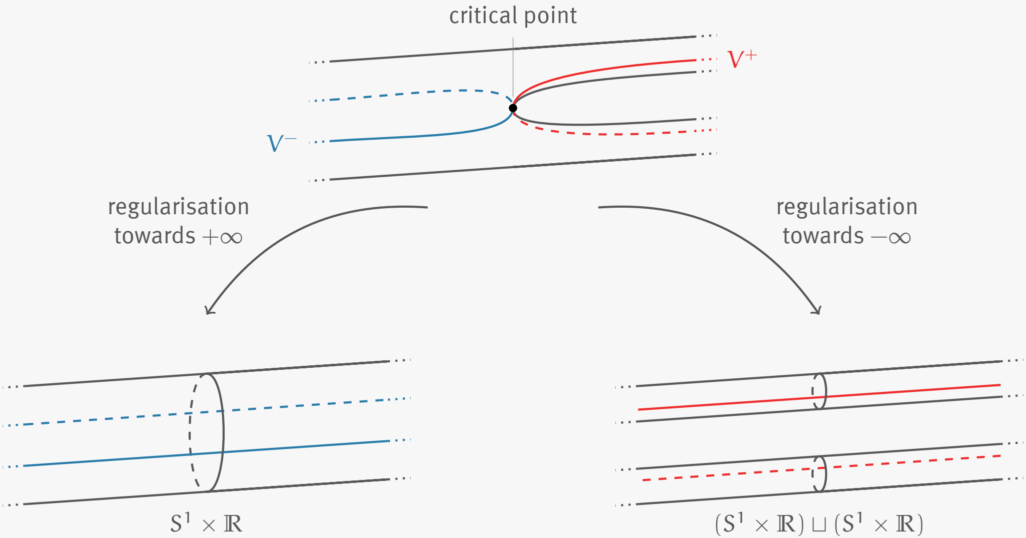
Plots of the saddle





Regularisation involves choices!



critical point

V^+

V^-

regularisation
towards $+\infty$

regularisation
towards $-\infty$

$S^1 \times \mathbb{R}$

$(S^1 \times \mathbb{R}) \sqcup (S^1 \times \mathbb{R})$

Custom indexing category

Definition

Let $\mathcal{K}^{[k]}$ be the category of

- finite sets T equipped with labelling functions $T \to \{k, \dots, d-k\}$
- Morphisms are injections over $\{k, \dots, d-k\}$ plus signs ± 1 for all points not in the image.

Definition

The $\mathcal{L}_T^{[k]}$ contains the same data as $\mathcal{L}^{[k]}$ plus a choice ± 1 of regularisation direction for all but finitely many critical points, which instead are indexed by $T \in \mathcal{K}^{[k]}$.

Lemma

$$\operatorname*{hocolim}_{T \in \mathcal{K}^{[k]}} \mathcal{L}_T^{[k]} \xrightarrow{\simeq} \mathcal{L}^{[k]} \xrightarrow{\simeq} \mathcal{D}^{[k]}$$

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Regularisation and $(d-1)$-mainfolds with surgery data

Fix T and an element in $\mathcal{L}_T^{[k]}$

- Move critical points indexed by T to height zero and others to height $\leqslant -1$ or $\geqslant +1$ resp.
- Perform all regularisations (regularise the critical points indexed by T towards $+\infty$) $\rightsquigarrow$ new height function f^{rg}.
- The embedded saddles indexed by T give surgery data $S^p \times D^{d-1-p} \hookrightarrow (f^{rg})^{-1}(0)$ for $p \in \{k-1, \dots, d-k-1\}$

$\Rightarrow$ get a $(d-1)$-manifold $(f^{rg})^{-1}(0)$ equipped with surgery data indexed by T

Definition

Let $\mathcal{W}_T^{[k]}$ be the space of closed $(d-1)$-manifolds M equipped with surgery data indexed by T and $\ell\colon M \to BO(d)$ $(k-1)$-connected.

Lemma

The above procedure defines a map, which is a levelwise weak equivalence

$$\operatorname*{hocolim}_{T\in\mathcal{K}^{[k]}} \mathcal{W}_T^{[k]} \xleftarrow{\simeq} \operatorname*{hocolim}_{T\in\mathcal{K}^{[k]}} \mathcal{L}_T^{[k]}$$

Homotopy type of the cobordism category with Morse functions

Definition (Morse Grassmannian)

For integers k and N we let $\mathrm{Gr}^{[k]}_{d,\theta}(\mathbb{R}^{d+N})$ denote the space of triples (V, l, σ) where

(i) $V \subset \mathbb{R}^{d+N}$ is an element of $\mathrm{Gr}_{d,\theta}(\mathbb{R}^{d+N})$

(ii) $l\colon V \to \mathbb{R}$ linear functional and $\sigma\colon V \times V \to \mathbb{R}$ symmetric bilinear form s.th.: If $l = 0$, then σ is non-degenerate with $k \leqslant \mathrm{index}(\sigma) \leqslant d - k$

As for the usual Grassmannian: Build a Thom spectrum out of the Thom spaces of the complements of the canonical bundles $\gamma_\theta \to \mathrm{Gr}^{[k]}_{d,\theta}$:

$$\mathrm{hW}^{[k]}_{d,\theta} \coloneqq \mathrm{Th}(-\gamma_\theta)$$

Theorem [MW07; Per17a]

The Pontryagin–Thom construction yields weak equivalences

$$\Omega^{\infty-1}\mathrm{hW}^{[k]}_{d,\theta} \simeq \mathcal{D}^{[k]}_\theta \simeq B\mathrm{Cob}^{\mathrm{mf},k}_\theta$$

The proof is a very involved inductive argument in k with [MW07, Thm. 1.2] as base case.

Proofsketch

Recap of parametrised Morse theory

$$\begin{tikzcd}
\mathcal{W}^{[k]}_{\theta,\emptyset} \arrow[r, hook] & \operatorname*{hocolim}_{T\in\mathcal{K}^{[k]}} \mathcal{W}^{[k]}_{\theta,T} & \operatorname*{hocolim}_{T\in\mathcal{K}^{[k]}} \mathcal{L}^{[k]}_{\theta,T} \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & \mathcal{D}^{[k]}_{\theta} \arrow[r, hook] \arrow[d, phantom, sloped, "\simeq"] & \mathcal{D}_{\theta,1} \arrow[d, phantom, sloped, "\simeq"] \\
 & & \Omega^{\infty}\mathrm{MT}\theta_{d-1} \arrow[r] & \Omega^{\infty-1}\mathrm{hW}^{k}_{\theta,d} \arrow[r] & \Omega^{\infty-1}\mathrm{MT}\theta_{d}
\end{tikzcd}$$

- There are comparison maps $\Sigma^{-1}\mathrm{MT}\theta_{d-1} \to \mathrm{hW}^{k}_{\theta,d} \to \mathrm{MT}\theta_{d}$.
- $\mathcal{W}^{[k]}_{\theta,\emptyset}$ is the space of closed $(d-1)$-manifolds with θ_{d-1}-structure, where θ_{d-1} is the restriction of θ to $BO(d-1)$ such that the structure map is $(k-1)$-connected

Index theory for spaces of manifolds

Specialize the tangential structure to $\theta\colon \mathrm{BSpin}(d) \times BG \to BO(d)$ and set $C^*(\theta_d) \coloneqq C^*(G) \otimes C\ell_d$

$$\begin{tikzcd}
\mathcal{D}_\theta^{[k],\mathrm{psc}} \arrow[r, hook] \arrow[d] & \mathcal{D}_\theta^{\mathrm{psc}} \arrow[r] \arrow[d, "F"] & \mathbb{D}(C^*(\theta_d))_1 \arrow[d, hook] \arrow[r, phantom, "\simeq"] & * \\
\mathcal{D}_\theta^{[k]} \arrow[r, hook] & \mathcal{D}_{\theta,1} \arrow[r, "\mathrm{ind}_1"] & \mathbb{KO}(C^*(\theta_d))_1 &
\end{tikzcd}$$

- Ebert [Ebe19] established an index theory for spaces of manifolds (in the generality of C^*-linear Dirac operators).
- Roughly: Index as map of spectra from the spaces of manifolds spectrum of Galatius and Randal-Williams [GRW10].

Lemma

- *The composition* $F \circ \mathrm{ind}_1$ *factors through degenerate* KK*-cycles.*
- *The space of degenerate cycles is contractible*

The fibration theorem

Definition

We define $\mathcal{W}_{\theta,T}^{[k],\mathrm{psc}}$ as $\mathcal{W}_{\theta,T}^{[k]}$ plus a psc metric g on M, which restricts to the standard metric on the surgery data, i.e. $e^*g = g_{\mathrm{round}} + g_{\mathrm{tor}}$ for $e\colon S^p \times D^{d-k} \hookrightarrow M$.

Theorem

For $k \geqslant 3$ the forgetful map

$$\operatorname*{hocolim}_{T \in \mathcal{K}^{[k]}} \mathcal{W}_{\theta,T}^{[k],\mathrm{psc}} \to \operatorname*{hocolim}_{T \in \mathcal{K}^{[k]}} \mathcal{W}_{\theta,T}^{[k]}$$

is a quasifibration with fibre over $(M, \emptyset)$ given by $\mathcal{R}^+(M)$.

- The assumption $k \geqslant 3$ arises from the Gromov–Lawson–Chernysh surgery. Here the symmetric restriction of the Morse indices fits perfectly!
- $k \geqslant 3 \implies d \geqslant 6$
- The tangential 2-type of a spin manifold M^n is $\mathrm{BSpin}(n) \times B\pi_1 M \to \mathrm{BO}(n)$, hence in our case of interest $(k-1)$-connectivity of the structure maps is not an issue for $k = 3$.

After finding psc variants for $\mathcal{L}_{\theta,T}^{[k]}$ and an identification of the induced map on homotopy fibres...

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The main diagram

For $d \geqslant 6$, $\dim M = d-1$ and $k = 3$

$$\begin{tikzcd}
& \mathcal{R}^+(M) \arrow[rrrr, "\mathrm{inddiff}^G"] \arrow[d] & & & & \Omega\mathbb{KO}(C^*(\theta_d))_1 \arrow[d] \\
\mathcal{W}^{[k],\mathrm{psc}}_{\theta,\emptyset} \arrow[r, hook] \arrow[d] & \mathcal{W}^{[k],\mathrm{psc}}_{\theta} \arrow[d] & \operatorname*{hocolim}_{T\in\mathcal{K}^{[k]}} \mathcal{L}^{[k],\mathrm{psc}}_{\theta,T} \arrow[l, "\simeq"'] \arrow[r] \arrow[d] & \mathcal{D}^{[k],\mathrm{psc}}_{\theta} \arrow[r, hook] \arrow[d] & \mathcal{D}^{\mathrm{psc}}_{\theta} \arrow[r] \arrow[d, "F"] & \mathbb{D}(C^*(\theta_d))_1 \simeq * \arrow[d, hook] \\
\mathcal{W}^{[k]}_{\theta,\emptyset} \arrow[r, hook] & \mathcal{W}^{[k]}_{\theta} & \operatorname*{hocolim}_{T\in\mathcal{K}^{[k]}} \mathcal{L}^{[k]}_{\theta,T} \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & \mathcal{D}^{[k]}_{\theta} \arrow[r, hook] \arrow[d, phantom, "\wr"] & \mathcal{D}_{\theta,1} \arrow[r, "\mathrm{ind}_1"] \arrow[d, phantom, "\wr"] & \mathbb{KO}(C^*(\theta_d))_1 \\
& & \Omega^{\infty}MT\theta_{d-1} \arrow[r] \arrow[rrru, bend right] & \Omega^{\infty-1}hW^{k}_{\theta,d} \arrow[r] \arrow[rru, bend right] & \Omega^{\infty-1}MT\theta_d \arrow[ru, bend right] &
\end{tikzcd}$$

Commutativity of maps to $\mathbb{KO}(C^*(\theta_d))_1$ follows from index theorem

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Corollary: Factorisation

Theorem

There is a fibration p *with fibre* $\mathcal{R}^+(M)$ *such that the diagram on the right is homotopy commutative and the induced map on homotopy fibres is* $\mathrm{inddiff}^G$.

$$\begin{tikzcd}
 & \mathcal{X} \arrow[r] \arrow[d, "p"] & * \arrow[d] \\
\Omega^\infty MT\theta_{d-1} \arrow[r] & \Omega^{\infty-1} hW^k_{\theta,d} \arrow[r, "\Omega^{\infty-1}\eta_d"] & \mathbb{K}\mathrm{O}(C^*(\theta_d))_1
\end{tikzcd}$$

Taking the fibre transport of p at g_0 yields ρ and the following factorisation for $d-1 \geqslant 5$

$$\begin{tikzcd}
\Omega^{\infty+1} MT\theta_{d-1} \arrow[r] \arrow[rrr, bend right, "\Omega^{\infty+1}\eta_{d-1}"'] & \Omega^\infty hW^k_{\theta,d} \arrow[r, "\rho"] & \mathcal{R}^+(M)_h \arrow[r, "\mathrm{inddiff}^G_{g_0}"] & \Omega^{\infty+d}\mathbb{K}\mathrm{O}(C^*(G))
\end{tikzcd}$$

Remark: Similar diagrams are used in [BERW17; ERW19a; ERW19b]

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The action $\mathrm{Diff}(M) \curvearrowright \mathcal{R}^+(M)$

$$\begin{tikzcd}
\mathcal{R}^+(M) \arrow[r, equal] \arrow[d] & \mathcal{R}^+(M) \arrow[r, equal] \arrow[d] & \mathcal{R}^+(M) \arrow[d] \\
\mathcal{R}^+(M) /\!\!/ \mathrm{Diff}^\theta(M) \arrow[r] \arrow[d] & \mathcal{W}^{[3],\mathrm{psc}}_{\theta,\emptyset} \arrow[r, hook] \arrow[d] & \operatorname*{hocolim}_{T \in \mathcal{K}^{[k]}} \mathcal{W}^{[3],\mathrm{psc}}_{\theta,T} \arrow[d] \\
\mathrm{BDiff}^\theta(M) \arrow[r, hook] & \mathcal{W}^{[3]}_{\theta,\emptyset} \arrow[r, hook] & \operatorname*{hocolim}_{T \in \mathcal{K}^{[k]}} \mathcal{W}^{[3]}_{\theta,T} \arrow[r, phantom, "\simeq"] & \Omega^{\infty-1} hW^3_{\theta,d}
\end{tikzcd}$$

Theorem (B.)

$M^{d-1} \in \mathcal{W}^{[3],\mathrm{psc}}_{\theta,\emptyset}$ simply connected (i.e. $\theta = \mathrm{BSpin}$) and $g_0 \in \mathcal{R}^+(M)$. The orbit map $\sigma_{g_0} :\ \mathrm{Diff}^{\mathrm{Spin}}(M) \to \mathcal{R}^+(M)$ factors in π_k through a map

$$\pi_k(\mathrm{Diff}^{\mathrm{Spin}}(M)) \simeq \pi_{k+1}(\mathrm{BDiff}^{\mathrm{Spin}}(M)) \longrightarrow \pi_{k+1}(MT\mathrm{Spin}(d-1))$$

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