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Spaces of PSC metrics and parametrised Morse theory

Detection theorems for $d \geqslant 5$ and higher index theory

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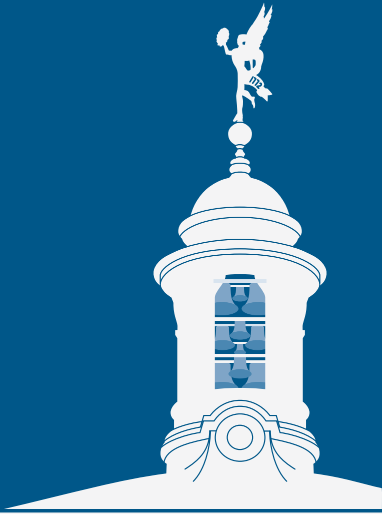

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Recap on positive scalar curvature metrics

Let (M, g) be a Riemannian manifold.

- Scalar curvature is the weakest curvature notion derived from the Riemannian curvature tensor.
- At each point $p \in M$ it is only a scalar quantity $\implies$ smooth function $\mathrm{scal}_g : M \to \mathbb{R}$
- KAZDAN and WARNER [KW75]: metrics of negative scalar curvature always exist

Existence question

Given a smooth manifold M. Does M admit a metric with $\mathrm{scal}_g > 0$?

- By now you probably know, that there are many known obstructions to admitting psc.
- Index theory provides the stronger results, but only works in the presence of spin structures.

The space of psc metrics

Definition (Space of metrics)

Let $\mathcal{R}(M) \subset C^\infty(M, \mathrm{Sym}^2(T^*M))$ be the **space of Riemannian metrics** (with the C^∞-topology).

Note: $\mathcal{R}(M)$ is convex, so $\mathcal{R}(M) \simeq *$

Definition (Space of psc metrics)

$$\mathcal{R}^+(M) := \{g \in \mathcal{R}(M) \mid \mathrm{scal}_g > 0\}$$

If M has boundary, one requires $g = h + dt^2$ near the boundary for $h \in \mathcal{R}^+(\partial M)$.

$$\mathcal{R}^+(M)_h := \{g \in \mathcal{R}^+(M) \mid g|_{\partial M} \equiv h\}$$

Uniqueness question

Assume $\mathcal{R}^+(M) \neq \emptyset$. What is the homotopy type of $\mathcal{R}^+(M)$?

First main tool: Gromov–Lawson–Chernysh surgery

Let $\phi\colon V^k \times \mathbb{R}^{d-k} \hookrightarrow W^d$ be an open embedding and $h_V \in \mathcal{R}^+(V)$

Definition ($\mathcal{R}^+$ with prescribed metrics)

$$\mathcal{R}^+(W,\phi) \coloneqq \{h \in \mathcal{R}^+(W) \mid \phi^* h = h_V + g_{\text{tor}}^{d-k} \text{ near } V \times D^{d-k} \subset V \times \mathbb{R}^{d-k}\}$$

Theorem (Chernysh [Che04])

If $d-k \geqslant 3$, then the inclusion

$$\mathcal{R}^+(W,\phi) \longrightarrow \mathcal{R}^+(W)$$

is a weak equivalence. Similarly for $\mathcal{R}^+(W)_g$

Corollary (Surgery equivalence)

For $V = S^k$ and $d-k, k \geqslant 3$ there is a preferred class of weak homotopy equivalences

$$\mathcal{R}^+(W)_g \simeq \mathcal{R}^+(W_\phi)_g$$

where W_ϕ denotes the surgered manifold.

Second main tool: the index difference

From now on we assume all manifolds to be spin.

Idea due to Hitchin [Hit74]

Define a map to K-theory using index theory and find elements of $\pi_k(\mathcal{R}^+(M)_h)$ that survive.

- Index of the Dirac operator D_g will be zero for every $g \in \mathcal{R}^+(M)_h$.
- Compare two metrics instead: $tg_0 + (1-t)g_1$ yields a path of Fredholm operators.
- Start and end are invertible for $g_0, g_1 \in \mathcal{R}^+(M)_h$
- After taking the index, the path can be interpreted as a loop in K-theory

(in a proper implementation of this idea one uses KK-cycles)

Definition (Index difference)

By fixing $g_0 \in \mathcal{R}^+(M)_h$

$$\mathrm{inddiff}_{g_0} : \mathcal{R}^+(M)_h \longrightarrow \Omega^{\infty+d+1}\mathbb{KO}$$

The detection theorem

Theorem (Botvinnik, Ebert, and Randal-Williams [BERW17])

M^d *compact spin,* $d \geqslant 6$, $h \in \mathcal{R}^+(\partial M)$, $g_0 \in \mathcal{R}^+(M)_h$ *and* $k \geqslant 0$

$$(\mathrm{inddiff}_{g_0})_* \colon \pi_k(\mathcal{R}^+(M)_h) \longrightarrow KO_{k+d+1=:m}(*) = \begin{cases} \mathbb{Z} & \textit{if } m \equiv 0 \mod 4 \\ \mathbb{Z}/2 & \textit{if } m \equiv 1,2 \mod 8 \\ 0 & \textit{else} \end{cases}$$

is (rationally) surjective.

Technical heart of the detection theorem

Let $\mathrm{MTSpin}(d)$ be the Madsen–Tillmann spectrum, $\lambda_{-d}\colon \mathrm{MTSpin}(d) \to \Sigma^{-d}\mathbb{KO}$ the KO-orientation and $\widehat{\mathcal{A}} = \Omega^\infty(\lambda_{-d})$

Theorem

There exists a fibration p fitting into the following diagram such that

(i) *the left square is homotopy cartesian*

(ii) *the induced map on homotopy fibres is* $\mathrm{inddiff}_{g_0}$

$$\begin{array}{ccccc}
\mathcal{R}^+(M)_h/\!\!/\,\mathrm{Diff}_\partial(M) & \longrightarrow & X & \longrightarrow & * \\
\downarrow & & \downarrow{\scriptstyle p} & & \downarrow \\
B\mathrm{Diff}_\partial(M) & \xrightarrow{\alpha_M} & \Omega^\infty \mathrm{MTSpin}(d) & \xrightarrow{\widehat{\mathcal{A}}} & \Omega^{\infty+d}\mathbb{KO}
\end{array}$$

After taking the fibre transport of p at g_0 to get ρ, the diagram implies that

$$\Omega^{\infty+1}\mathrm{MTSpin}(d) \xrightarrow{\rho} \mathcal{R}^+(M^d)_h \xrightarrow{\mathrm{inddiff}_{g_0}} \Omega^{\infty+d+1}\mathbb{KO}$$

is homotopic to $\Omega\widehat{\mathcal{A}}$. (Rational) surjectivity of $\widehat{\mathcal{A}}_* \implies$ Detection Theorem.

Improving the detection theorem

The Index of the Rosenberg-Dirac operator

- The spin Dirac operator on M can be twisted by a bundle E, which introduces an extra term in the Lichnerowicz formula (see enlargeability and Llarull's theorem).
- Rosenberg: Twist with the flat **Miščenko–Fomenko line bundle** $\mathcal{L}_G \coloneqq EG \times_G C^*(G) \to BG$, where $G = \pi_1 M$.
- The resulting twisted Dirac operator $D_{\mathcal{L}_G}$ now acts on a Hilbert-$C^*(G)$-module $\implies$ its index is defined by a $KK(C\ell^d, C^*(G))$-cycle yielding an element in $KO_d(C^*(G))$
- By using the Rosenberg index we can generalise the index difference

$$\mathrm{inddiff}^G_{g_0} : \mathcal{R}^+(M)_h \longrightarrow \Omega^{\infty+d+1}\mathbb{KO}(C^*(G))$$

$\Rightarrow$ Potential to detect more homotopy classes of $\mathcal{R}^+(M)_h$!

- Analogue for the map $\widehat{\mathcal{A}}$:

$$\eta\colon \mathrm{MTSpin}(d) \wedge BG_+ \xrightarrow{\lambda_{-d}\wedge \mathrm{id}} \Sigma^{-d}\mathbb{KO} \wedge BG_+ \xrightarrow{\Sigma^{-d}\nu} \Sigma^{-d}\mathbb{KO}(C^*(G))$$

where ν is the Novikov assembly map.

The Detection theorem for higher index theory

EBERT and RANDAL-WILLIAMS [ERW19a; ERW19b] haven proven a detection theorem involving $G = \pi_1 M$.The technical heart in this case is:

Theorem (Technical heart in [ERW19b])

M^d spin, compact, $d \geqslant 6$ and $(M, \partial M)$ 2-connected. Then the technical heart theorem holds for

$$\begin{array}{ccccc} \mathcal{R}^+(M)_h /\!\!/ \operatorname{Diff}_\partial(M) & \longrightarrow & X & \longrightarrow & * \\ \downarrow & & \downarrow p & & \downarrow \\ B\operatorname{Diff}_\partial(M) & \xrightarrow{\alpha_M} & \Omega^\infty MT\operatorname{Spin}(d) \wedge BG_+ & \xrightarrow{\eta} & \Omega^{\infty+d}\mathbb{KO}(C^*(G)) \end{array}$$

Actual detection results are harder to state, because of the involved assembly map. In particular, they will always need additional assumptions on G.

Extension to $d \geqslant 5$

Theorem (Perlmutter [Per17b])

The original detection theorem also holds for $d \geqslant 5$.

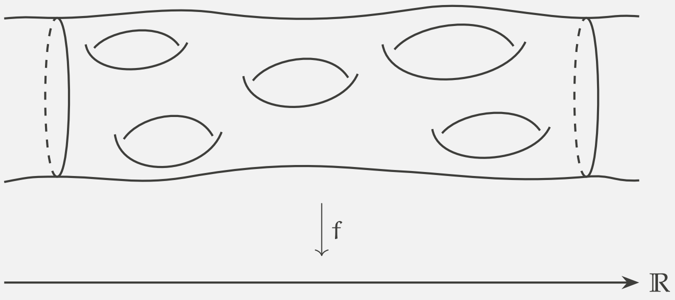
- Back to the roots: Replace GRW methods by original methods of Madsen and Weiss [MW07]
- Series of two preprints (sadly, he left mathematics):
    1. Extension of Madsen and Weiss [MW07] methods (to reprove high-dimensional MW theorem of Galatius and Randal-Williams [GRW14])
    2. Application to PSC

Theorem (B.)

Both improvements can be carried out in unison.

Parametrised Morse Theory

Spaces of manifolds: Long manifolds



Definition

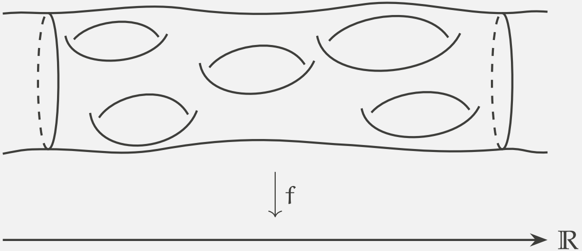
Let $\theta\colon B \to BO(d)$ be a fibration, e.g. $BSpin(d) \to BO(d)$. The **space of θ-manifolds with one non-compact direction** is given by

$$\mathcal{D}_{\theta,1} = \{(W, f) \mid f\colon W^d \to \mathbb{R} \text{ smooth and proper}\}$$

Theorem ([GMTW09; GRW10])

$$\Omega^{\infty-1}MT\theta \simeq \mathcal{D}_{\theta,1} \simeq BCob_\theta$$

How can we turn a long d-manifold into a $(d-1)$-manifold?



Just take $f^{-1}(a)$ at a regular value $a \in \mathbb{R}$!

- MADSEN and WEISS method: Non-destructive way to lower the dimension like this!
- Have to perform a "regularisation" to avoid critical points

Definition

Let $0 \leqslant k \leqslant \lfloor d/2 \rfloor$.
Let $\mathcal{D}_\theta^{[k]} \subset \mathcal{D}_{\theta,1}$ subspace with f Morse, Morse indices in $\{k, \ldots, d-k\}$ and $\ell\colon W \to B$ $(k-1)$-connected.

The restriction on Morse indices was introduced by PERLMUTTER [Per17a].

Local model for critical points

Definition

$V = V^+ \oplus V^-$ inner product space. The **saddle** is defined as

$$\mathrm{sdl}(V) := \left\{ v \in V \,\middle|\, \|v_+\|^2 \|v_-\|^2 \leqslant 1 \right\}$$

Canonical height function on $\mathrm{sdl}(V)$ with unique critical point at the origin:

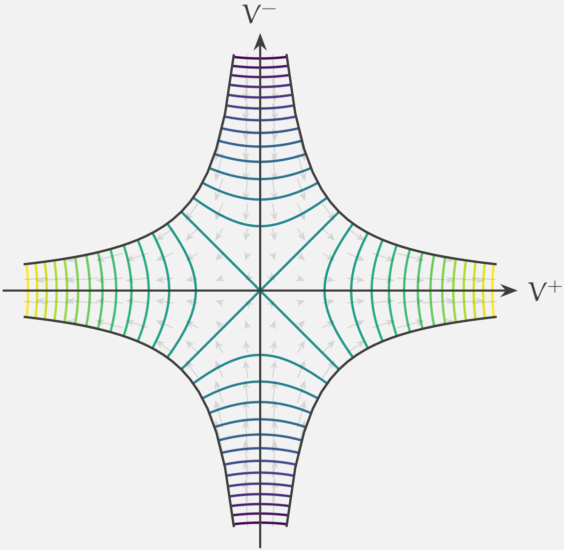
$$f_V(v) = \|v_+\|^2 - \|v_-\|^2$$

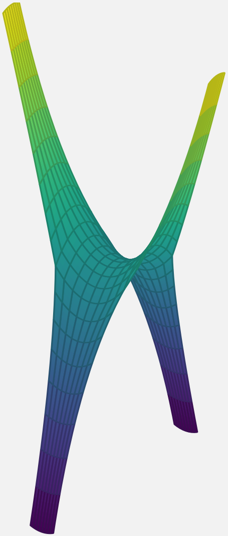
Regularisation: Remove V^+ or V^- and adjust height function such that height approaches $+\infty$ near V^+.

Definition

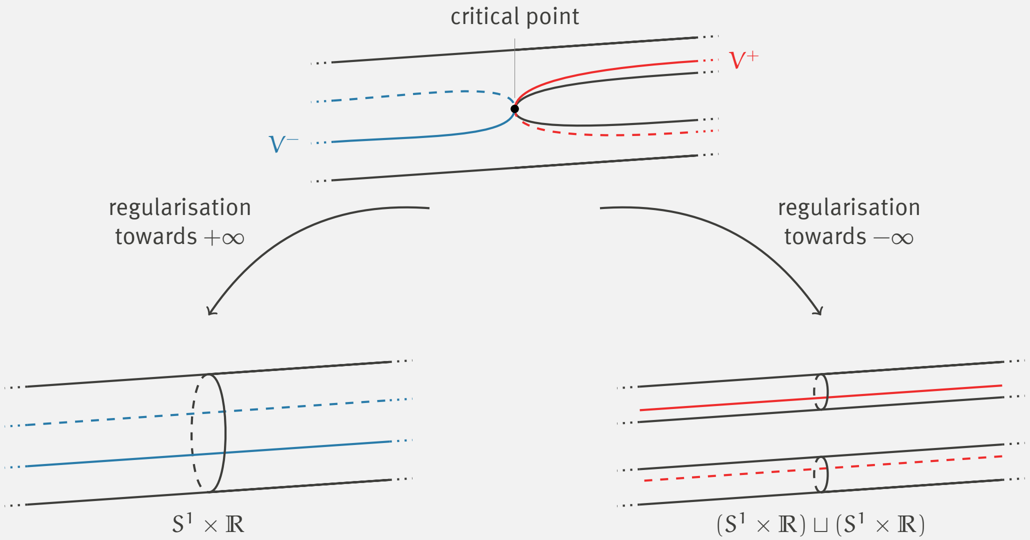
$\mathcal{L}_\theta^{[k]}$ has the same data as $\mathcal{D}_\theta^{[k]}$ with embedded saddles around all critical points, such that the height functions f_V and f are compatible.

Plots of the saddle





Regularisation involves choices!



Custom indexing category

Definition

Let $\mathcal{K}^{[k]}$ be the category of

- finite sets T equipped with labelling functions $T \to \{k, \ldots, d-k\}$
- Morphisms are injections over $\{k, \ldots, d-k\}$ plus signs ± 1 for all points not in the image.

Definition

The $\mathcal{L}_{\theta,T}^{[k]}$ contains the same data as $\mathcal{L}_{\theta}^{[k]}$ plus a choice ± 1 of regularisation direction for all but finitely many critical points, which instead are indexed by $T \in \mathcal{K}^{[k]}$.

Lemma

$$\operatorname*{hocolim}_{T\in\mathcal{K}^{[k]}} \mathcal{L}_{\theta,T}^{[k]} \xrightarrow{\simeq} \mathcal{L}_{\theta}^{[k]} \xrightarrow{\simeq} \mathcal{D}_{\theta}^{[k]}$$

$(d-1)$-mainfolds with surgery data

Definition

Let $\mathcal{W}^{[k]}_{\theta,T}$ be the space of closed $(d-1)$ θ-manifolds M equipped with surgery data indexed by T and $\ell\colon M \to B$ $(k-1)$-connected.

Given an element in $\mathcal{L}^{[k]}_{\theta,T}$

- Move critical points indexed by T to height zero and others to height $\leqslant -1$ or $\geqslant +1$ resp.
- perform all regularisations (regularise the critical points indexed by T towards $+\infty$) $\rightsquigarrow$ new height function f^{rg}.
- The embedded saddles indexed by T give surgery data $S^p \times D^{d-k} \hookrightarrow (f^{rg})^{-1}(0)$ for $p \in \{k-1, \dots, d-k-1\}$

$\Rightarrow$ get an element of $\mathcal{W}^{[k]}_{\theta,T}$

Lemma

The above procedure defines a map, which is a levelwise weak equivalence

$$\operatorname*{hocolim}_{T\in\mathcal{K}^{[k]}} \mathcal{W}^{[k]}_{\theta,T} \xleftarrow{\simeq} \operatorname*{hocolim}_{T\in\mathcal{K}^{[k]}} \mathcal{L}^{[k]}_{\theta,T}$$

Homotopy type of the cobordism category with Morse functions

Definition (Morse Grassmannian)

For integers k and N we let $\mathrm{Gr}^{[k]}_{\theta,d}(\mathbb{R}^{d+N})$ denote the space of triples (V, l, σ) where

(i) $V \subset \mathbb{R}^{d+N}$ is an element of $\mathrm{Gr}_{\theta,d}(\mathbb{R}^{d+N})$

(ii) $l\colon V \to \mathbb{R}$ linear functional and $\sigma\colon V \times V \to \mathbb{R}$ symmetric bilinear form s.th.: If $l = 0$, then σ is non-degenerate with $k \leqslant \mathrm{index}(\sigma) \leqslant d - k$

As for the usual Grassmannian: Build a Thom spectrum out of the Thom spaces of the complements of the canonical bundles $\gamma_\theta \to \mathrm{Gr}^{[k]}_{\theta,d}$:

$$hW^k_{\theta,d} = \mathrm{Th}(-\gamma_\theta)$$

Theorem ([MW07; Per17a])

The Pontryagin–Thom construction yields weak equivalences

$$\Omega^{\infty-1} hW^k_{\theta,d} \simeq \mathcal{D}^{[k]}_\theta \simeq B\mathrm{Cob}^{mf,k}_\theta$$

The proof is a very involved inductive argument in k with [MW07, Thm. 1.2] as base case.

Proofsketch

Recap of parametrised Morse theory

$$\begin{array}{ccccccccc}
\mathcal{W}_{\theta,\emptyset}^{[k]} & \hookrightarrow & \operatorname*{hocolim}_{T\in\mathcal{K}^{[k]}} \mathcal{W}_{\theta,T}^{[k]} & \xleftarrow{\simeq} & \operatorname*{hocolim}_{T\in\mathcal{K}^{[k]}} \mathcal{L}_{\theta,T}^{[k]} & \xrightarrow{\simeq} & \mathcal{D}_{\theta}^{[k]} & \hookrightarrow & \mathcal{D}_{\theta,1} \\
\| & & & & & & \wr\| & & \wr\| \\
\coprod_{[M]} B\mathrm{Diff}^{\theta_{d-1}}(M) & & \xrightarrow{\alpha} & & \Omega^{\infty}MT\theta_{d-1} & \longrightarrow & \Omega^{\infty-1}hW_{\theta,d}^{k} & \longrightarrow & \Omega^{\infty-1}MT\theta_{d}
\end{array}$$

- There are comparison maps $\Sigma^{-1}MT\theta_{d-1} \to hW_{\theta,d}^{k} \to MT\theta_{d}$.
- $\mathcal{W}_{\theta,\emptyset}^{[k]}$ is the space of closed $(d-1)$-manifolds with θ_{d-1}-structure, where θ_{d-1} is the restriction of θ to $BO(d-1)$.
- α is the parametrised Pontryagin–Thom map

Index theory for spaces of manifolds

Specialize the tangential structure to $\theta\colon \mathrm{BSpin}(d) \times BG \to BO(d)$ and set $C^*(\theta_d) \coloneqq C^*(G) \otimes C\ell_d$

$$\begin{array}{ccccc}
\mathcal{D}_\theta^{[k],\mathrm{psc}} & \hookrightarrow & \mathcal{D}_\theta^{\mathrm{psc}} & \longrightarrow & \mathbb{D}(C^*(\theta_d))_1 \simeq * \\
\downarrow & & \downarrow F & & \downarrow \\
\mathcal{D}_\theta^{[k]} & \hookrightarrow & \mathcal{D}_{\theta,1} & \xrightarrow{\mathrm{ind}_1} & \mathbb{KO}(C^*(\theta_d))_1
\end{array}$$

- Ebert [Ebe19] established an index theory for spaces of manifolds (in the generality of C^*-linear Dirac operators).
- Roughly: Index as map of spectra from the spaces of manifolds spectrum of Galatius and Randal-Williams [GRW10].

Lemma

- *The composition $F \circ \mathrm{ind}_1$ factors through degenerate KK-cycles.*
- *The space of degenerate cycles is contractible*

The fibration theorem

Definition

We define $\mathcal{W}_{\theta,T}^{[k],\mathrm{psc}}$ as $\mathcal{W}_{\theta,T}^{[k]}$ plus a psc metric g on M, which restricts to the standard metric on the surgery data, i.e. $e^*g = g_{\mathrm{round}} + g_{\mathrm{tor}}$ for $e\colon S^p \times D^{d-k} \hookrightarrow M$.

Theorem

For $k \geqslant 3$ the forgetful map

$$\operatorname*{hocolim}_{T\in\mathcal{K}^{[k]}} \mathcal{W}_{\theta,T}^{[k],\mathrm{psc}} \to \operatorname*{hocolim}_{T\in\mathcal{K}^{[k]}} \mathcal{W}_{\theta,T}^{[k]}$$

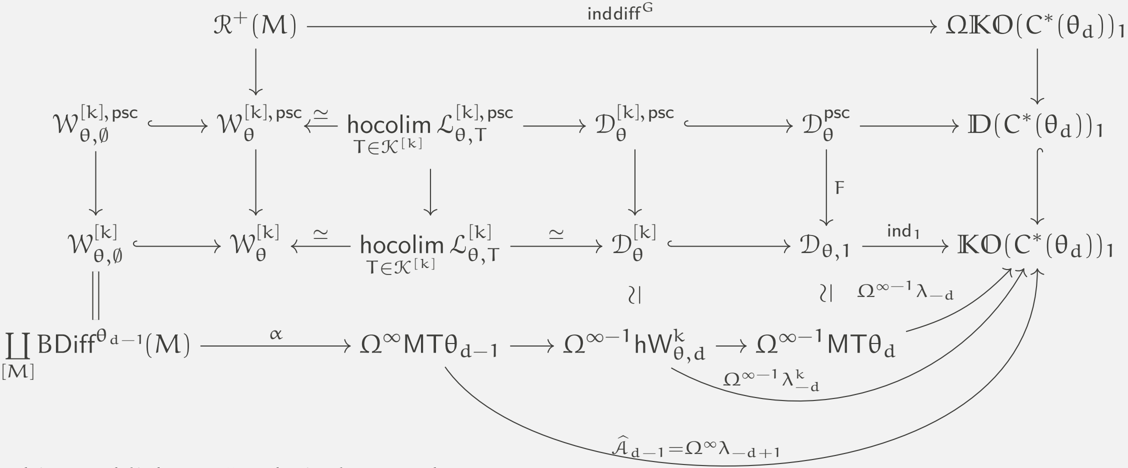
is a quasifibration with fibre over $(M,\emptyset)$ given by $\mathcal{R}^+(M)$.

- The assumption $k \geqslant 3$ arises from the Gromov–Lawson–Chernysh surgery. Here the symmetric restriction of the Morse indices fits perfectly!
- $k \geqslant 3 \implies d \geqslant 6$
- The tangential 2-type of a spin manifold M^n is $\mathrm{BSpin}(n) \times B\pi_1 M \to \mathrm{BO}(n)$, hence in our case of interest $(k-1)$-connectivity of the structure maps is not an issue for $k=3$.

After finding psc variants for $\mathcal{L}_{\theta,T}^{[k]}$ and an identification of the induced map on homotopy fibres...

Final diagram

For $d \geqslant 6$ and $k = 3$.



This establishes a “Technical Heart Theorem”!

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