Spaces of PSC metrics and parametrised Morse theory

A detection theorem for $d \geqslant 5$ and higher index theory

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Augsburg, June 5, 2023

Let (M, g) be a Riemannian manifold.

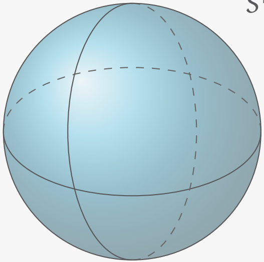
- Central notion in differential geometry: Riemannian curvature tensor associated to g.
- Tensor contraction turns this into **scalar curvature** $\rightsquigarrow$ smooth function $\mathrm{scal}_g \colon M \to \mathbb{R}$

Existence question

Given a smooth manifold M, does M admit a metric with $\mathrm{scal}_g > 0$?

- Admitting a positive scalar curvature metric has topological implications
- Only orientable surface admitting psc is S^2
- In fact, there are many (topological) obstructions to admitting psc, e.g. the $\widehat{A}$-genus

S^n



$$\mathrm{scal} \equiv \frac{n(n-1)}{r^2}$$

Dimension 2: Gauß–Bonnet

$$0 < \int_M \mathrm{scal}_g \; d\omega = 4\pi \cdot \chi(M)$$

Definition (Space of metrics)

Let $\mathcal{R}(M)$ be the **space of Riemannian metrics** with the C^∞-topology.

Note: $\mathcal{R}(M)$ is convex, so $\mathcal{R}(M) \simeq *$

Definition (Space of psc metrics)

$$\mathcal{R}^+(M) \coloneqq \left\{ g \in \mathcal{R}(M) \,\middle|\, \mathrm{scal}_g > 0 \right\}$$

If M has boundary, prescribe a metric h on ∂M and require product structure $\rightsquigarrow \mathcal{R}^+(M)_h$.

Uniqueness question

Assume $\mathcal{R}^+(M) \neq \emptyset$. What is the homotopy type of $\mathcal{R}^+(M)$?

First main tool: Gromov–Lawson–Chernysh surgery

Let $\phi\colon V^k \times \mathbb{R}^{d-k} \hookrightarrow M^d$ be an open embedding and $h_V \in \mathcal{R}^+(V)$

Definition

$$\mathcal{R}^+(M, \phi) \coloneqq \text{ space of psc metrics with prescribed metric on } \mathrm{im}(\phi)$$

Theorem (Chernysh [Che04])

If $d - k \geqslant 3$, then the inclusion

$$\mathcal{R}^+(M, \phi) \longrightarrow \mathcal{R}^+(M)$$

is a weak equivalence. Similarly for $\mathcal{R}^+(M)_h$

Corollary (Surgery equivalence)

For $V = S^k$ and $d - k, k + 1 \geqslant 3$ there is a preferred class of weak homotopy equivalences

$$\mathcal{R}^+(M) \simeq \mathcal{R}^+(M_\phi)$$

where M_ϕ denotes the surgered manifold.

Second main tool: the index difference

From now on we assume all manifolds to be spin.

Index difference of Hitchin [Hit74]

Idea: define a map to K-theory using index theory and use it to detect non-trivial elements of $\pi_k(\mathcal{R}^+(M)_h)$. Result: For $g_0 \in \mathcal{R}^+(M)$

$$\mathrm{inddiff}_{g_0} : \mathcal{R}^+(M) \longrightarrow \Omega^{\infty+d+1}\mathbb{KO}$$

- Index of the Dirac operator D_g will be zero for every $g \in \mathcal{R}^+(M)$ (Lichnerowicz).
- Compare two metrics instead: $tg_0 + (1-t)g_1$ yields a path of Fredholm operators.
- Start and end are invertible, since $g_0, g_1 \in \mathcal{R}^+(M)$
- The invertible operators make up a contractible space (Kuiper)

$\Rightarrow$ After taking the index, the path can be interpreted as a loop in K-theory

(in a proper implementation of this idea I would use KK-cycles)

Let M^d be compact spin, $d \geqslant 6$, $h \in \mathcal{R}^+(\partial M)$, $g_0 \in \mathcal{R}^+(M)_h$ and $k \geqslant 0$.

Theorem (Botvinnik, Ebert, and Randal-Williams [BERW17])

The induced map

$$(\mathrm{inddiff}_{g_0})_* \colon \pi_k\big(\mathcal{R}^+(M)_h\big) \longrightarrow KO_{k+d+1=:m}(*) = \begin{cases} \mathbb{Z} & \text{if } m \equiv 0 \mod 4 \\ \mathbb{Z}/2 & \text{if } m \equiv 1,2 \mod 8 \\ 0 & \text{else} \end{cases}$$

is (rationally) surjective.

The factorisation theorem

- Let $\mathrm{MTSpin}(d)$ be the Madsen–Tillmann spectrum
- There is a KO-orientation $\lambda_{-d}\colon \mathrm{MTSpin}(d) \to \Sigma^{-d}\mathbb{KO}$ ("**topological index**")

Theorem

There exists a map $\rho\colon \Omega^{\infty+1}\mathrm{MTSpin}(d) \to \mathcal{R}^+(M)_h$ *such that*

$$\begin{tikzcd} \Omega^{\infty+1}\mathrm{MTSpin}(d) \arrow[r, "\rho"] \arrow[rr, bend right, "\Omega^{\infty+1}\lambda_{-d}"'] & \mathcal{R}^+(M)_h \arrow[r, "\mathrm{inddiff}_{g_0}"] & \Omega^{\infty+d+1}\mathbb{KO} \end{tikzcd}$$

is homotopy commutative.

(Rational) surjectivity of $\Omega^{\infty+1}\lambda_{-d} \implies$ Detection Theorem.

Improving the detection theorem

Improvement 1: Higher index theory

“richer” target for index difference $\Longrightarrow$ Potential to detect more classes of $\pi_k \mathcal{R}^+(M)_h$

Higher index difference

$$\mathrm{inddiff}^G_{g_0} : \mathcal{R}^+(M)_h \longrightarrow \Omega^{\infty+d+1}\mathbb{KO}(C^*(G))$$

where $C^*(G)$ is the (reduced) group C^*-algebra for $G = \pi_1 M$.

- The spin Dirac operator on M can be twisted by a bundle E, which introduces an extra term in the Lichnerowicz formula (see enlargeability and Llarull’s theorem).
- Rosenberg: Twist with the flat **Miščenko line bundle** $\mathfrak{L}_G \coloneqq EG \times_G C^*(G) \to BG$
- twisted Dirac operator $D_{\mathfrak{L}_G}$ acts on a Hilbert-$C^*(G)$-module and has an index in $KO_d(C^*(G))$

Analogue for the topological index, using the Novikov assembly map ν:

$$\mathrm{topind}_G : MTSpin(d) \wedge BG_+ \xrightarrow{\lambda_{-d} \wedge \mathrm{id}} \Sigma^{-d}\mathbb{KO} \wedge BG_+ \xrightarrow{\Sigma^{-d}\nu} \Sigma^{-d}\mathbb{KO}(C^*(G))$$

Theorem (Ebert and Randal-Williams [ERW19; ERW22])

M^d spin, compact, $d \geqslant 6$. Then there exists ρ such that

$$\begin{tikzcd} \Omega^{\infty+1}(\mathrm{MTSpin}(d) \wedge BG_+) \arrow[r, "\rho"] \arrow[rr, bend right, "\Omega^{\infty+1}\,\mathrm{topind}_G"'] & \mathcal{R}^+(M)_h \arrow[r, "\mathrm{inddiff}^G_{g_0}"] & \Omega^{\infty+d+1}\mathbb{KO}(C^*(G)) \end{tikzcd}$$

is homotopy commutative

To derive detection results: study the assembly map $\rightsquigarrow$ assumptions on G.

Improvement 2: Extension to $d \geqslant 5$

Theorem (Perlmutter [Per17b])

The original detection and factorisation theorems also hold for $d \geqslant 5$.

- Back to the roots: Replace GRW methods by original methods of Madsen and Weiss [MW07]
- Series of two preprints (he left mathematics):
    1. Extension of Madsen and Weiss [MW07] methods
       (to reprove high-dimensional MW theorem of Galatius and Randal-Williams [GRW14])
    2. Application to PSC

Theorem B.

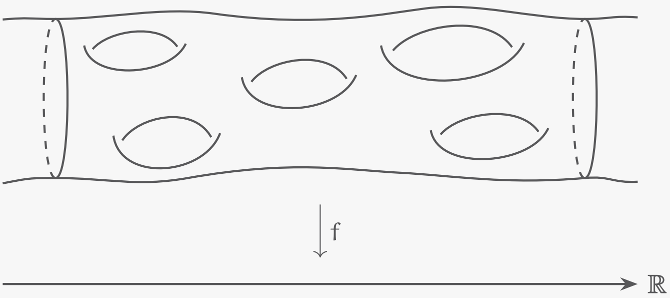
Both improvements, i.e.

1. incorporation the fundamental group via higher index theory
2. extension to $d \geqslant 5$

can be carried out in unison.

Parametrised Morse Theory

Spaces of manifolds: Long manifolds



Definition (Galatius and Randal-Williams)

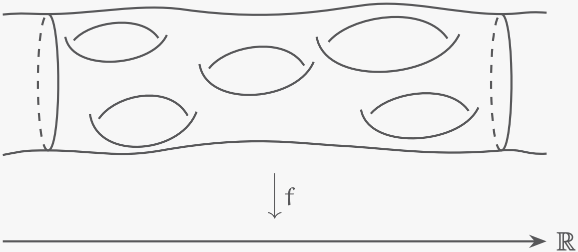
The **space of manifolds with one non-compact direction** is given by

$$\mathcal{D}_1 = \{(W, f) \mid f \colon W^{d+1} \to \mathbb{R} \text{ smooth and proper}\}$$

Theorem [GMTW09; GRW10]

$$\Omega^{\infty-1}\mathrm{MTO}(d+1) \simeq \mathcal{D}_1 \simeq \mathrm{BCob}_{d+1}$$

How can we turn a long $(d+1)$-manifold into a d-manifold?



Just take $f^{-1}(a)$ at a regular value $a \in \mathbb{R}$!

- Madsen and Weiss method: Non-destructive way to lower the dimension like this!
- Have to perform a "regularisation" to avoid critical points

Definition

Let $0 \leqslant k \leqslant \lfloor d + 1/2 \rfloor$.
Let $\mathcal{D}^{[k]} \subset \mathcal{D}_1$ subspace with f Morse, Morse indices in $\{k, \ldots, d+1-k\}$ and $\ell \colon W \to BO(d)$ $(k-1)$-connected.

The restriction on Morse indices was introduced by Perlmutter [Per17a].

Local model for critical points

Definition

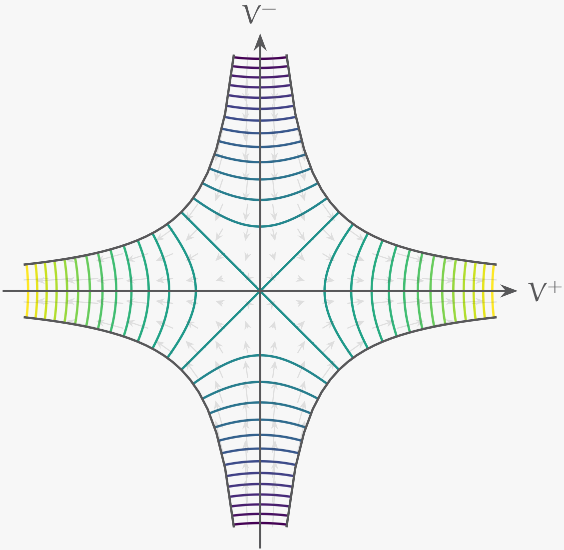
$V = V^+ \oplus V^-$ inner product space. The **saddle** is defined as

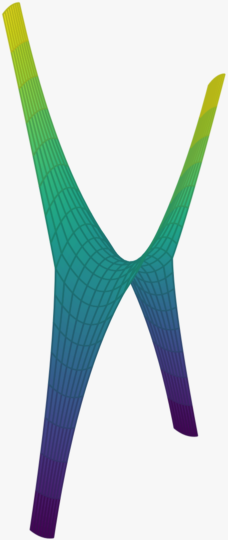
$$\mathrm{sdl}(V) \coloneqq \left\{ v \in V \,\middle|\, \|v_+\|^2 \|v_-\|^2 \leqslant 1 \right\}$$

Canonical height function on $\mathrm{sdl}(V)$ with unique critical point at the origin:

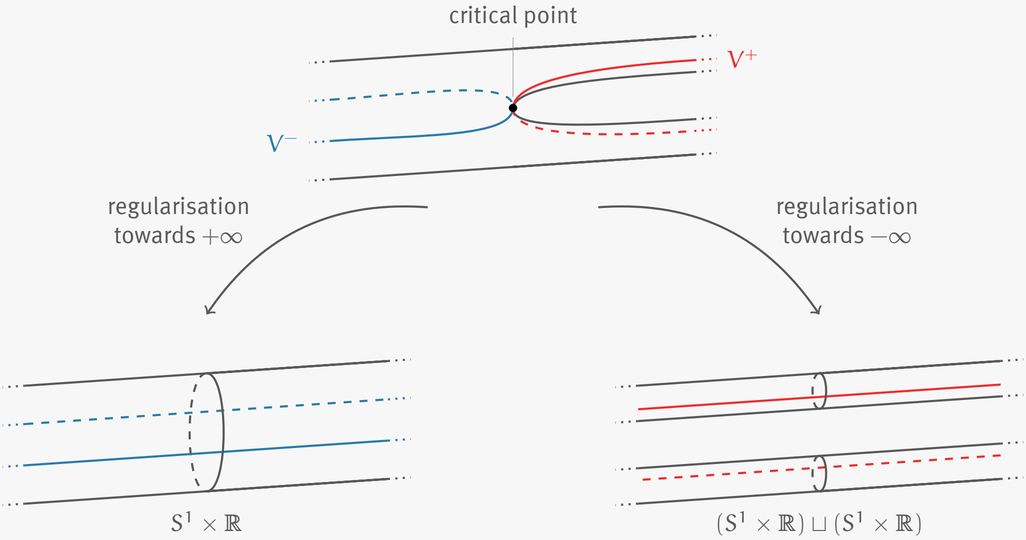
$$f_V(v) = \|v_+\|^2 - \|v_-\|^2$$

Plots of the saddle





Regularisation involves choices!



The homotopy colimit decomposition

For T a finite set:

Definition

Let $\mathcal{W}^{[k]}_T$ be the space of **closed** d-manifolds M equipped with surgery data indexed by T and $\ell \colon M \to BO(d)$ $(k-1)$-connected.

- $\mathcal{K}^{[k]}$ custom indexing category
  - finite sets and injections over $\{k, \ldots, d+1-k\}$
  - Morphisms know a sign $\pm\infty$ for elements not in the image

Theorem [MW07; Per17a]

$$\mathcal{D}^{[k]} \simeq \operatorname*{hocolim}_{T \in \mathcal{K}^{[k]}} \mathcal{W}^{[k]}_{\theta,T}$$

- Add local data at critical points
- Encode regularisation choices using $\mathcal{K}^{[k]}$
- Perform the regularisation and take preimage at zero.

Definition (Morse Grassmannian)

Let $Gr^{[k]}_{d+1,\theta}(\mathbb{R}^{d+1+N})$ denote the space of triples (V, l, σ) where

(i) $V \subset \mathbb{R}^{d+1+N}$ is an element of $Gr_{d+1,\theta}(\mathbb{R}^{d+1+N})$

(ii) $l\colon V \to \mathbb{R}$ linear functional and $\sigma\colon V \times V \to \mathbb{R}$ symmetric bilinear form s.th.: If $l = 0$, then σ is non-degenerate with $k \leqslant \mathrm{index}(\sigma) \leqslant d+1-k$

As for the usual Grassmannian: Build a Thom spectrum

$$MT\theta^{[k]} := Th(-\gamma_\theta)$$

where γ_θ is the canonical bundle.

Theorem [MW07; Per17a]

The Pontryagin–Thom construction yields weak equivalences

$$\Omega^{\infty-1} MT\theta^{[k]}(d+1) \simeq \mathcal{D}^{[k]}_\theta \simeq BCob^{mf,k}_\theta$$

- The proof is a very involved inductive argument in k with [MW07, Thm. 1.2] as base case.
- Homotopy colimit decomposition is central to the proof.

Proofsketch (back to psc)

Blueprint for the factorisation

Goal

Find a map ρ such that the following commutes

$$\begin{tikzcd} \Omega^{\infty+1}\mathrm{MTSpin}(d) \arrow[r, "\rho"] \arrow[rr, bend right, "\Omega^{\infty+1}\lambda_{-d}"'] & \mathcal{R}^+(M)_h \arrow[r, "\mathrm{inddiff}_{g_0}"] & \Omega^{\infty+d+1}\mathbb{KO} \end{tikzcd}$$

Idea

- Construct a fibration p with fibre $\mathcal{R}^+(M)$ as on the right
- ... fitting into the diagram on the right ...
- s.th. the induced map on homotopy fibres is $\mathrm{inddiff}_{g_0}$

The *fibre transport* of p yields

$$\rho\colon \Omega^{\infty+1}\mathrm{MTSpin}(d) \longrightarrow \mathcal{R}^+(M)$$

with the desired properties.

$$\begin{tikzcd} \mathcal{R}^+(M) \arrow[r] \arrow[d] & \Omega\mathbb{KO} \arrow[d] \\ \mathscr{X} \arrow[r] \arrow[d, "p"] & * \arrow[d] \\ \Omega^{\infty}\mathrm{MTSpin}(d) \arrow[r, "\Omega^{\infty}\lambda_{-d}"] & \mathbb{KO} \end{tikzcd}$$

Remark: Such diagrams are a common theme in all the previous work [BERW17; ERW19; ERW22].

Outline map of the proof

PSC fibration

The vanishing theorem

$$
\begin{tikzcd}
\mathcal{R}^+(M) \arrow[rrrr, "\mathrm{inddiff}^G"] \arrow[d] & & & & \Omega\mathbb{KO}(C^*_{d+1}(G))_1 \arrow[d] \\
\operatorname*{hocolim}_{T\in\mathcal{K}^{[k]}} \mathcal{W}^{[k],\mathrm{psc}}_{\theta,T} \arrow[r, dashed, leftrightarrow] \arrow[d] & \mathcal{D}^{[k],\mathrm{psc}}_{\theta} \arrow[r, hook] \arrow[d] & \mathcal{D}^{\mathrm{psc}}_{\theta} \arrow[d] & \mathcal{D}^{\mathrm{psc},\mathrm{op}}_{\theta} \arrow[l, "\simeq"'] \arrow[r] \arrow[d, "F"] & \mathbb{D}(C^*_{d+1}(G))_1 \simeq * \arrow[d, hook'] \\
\operatorname*{hocolim}_{T\in\mathcal{K}^{[k]}} \mathcal{W}^{[k]}_{\theta,T} \arrow[r, dashed, leftrightarrow, "\simeq"] & \mathcal{D}^{[k]}_{\theta} \arrow[r, hook] \arrow[d, phantom, "\wr|"] & \mathcal{D}_{\theta,1} \arrow[d, phantom, "\wr|"] & \mathcal{D}^{\mathrm{op}}_{\theta,1} \arrow[l, "\simeq"'] \arrow[r, "\mathrm{ind}_1"] & \mathbb{KO}(C^*_{d+1}(G))_1 \arrow[d, "\simeq"] \\
 & \Omega^{\infty-1}\mathrm{MT}\theta^{[k]} \arrow[r] & \Omega^{\infty-1}\mathrm{MT}\theta \arrow[rr, "\Omega^{\infty-1}\,\mathrm{topind}_G"] & & \Omega^{\infty+d}\mathbb{KO}(C^*(G))
\end{tikzcd}
$$

Parametrised Morse theory

Spaces of manifolds

Index theorem for spaces of manifolds

Corollary: Factorisation

Assumptions

- PSC fibration needs $k \geqslant 3 \implies d+1 \geqslant 6 \Leftrightarrow d \geqslant 5$
- Index theory needs the tangential structure $\theta\colon \mathrm{BSpin}(d) \times BG \to BO(d)$

Theorem (B.)

Let M be a compact connected spin manifold of dimension $d \geqslant 5$ with a map $f\colon M \to BG$ which is split-surjective on π_1. Furthermore, let $h \in \mathcal{R}^+(\partial M)$ and $g_0 \in \mathcal{R}^+(M)_h$. Then there is a map ρ such that the composition

$$\Omega^{\infty+1} MT\theta(d) \xrightarrow{\rho} \mathcal{R}^+(M)_h \xrightarrow{\mathrm{inddiff}^G_{g_0}} \Omega^{\infty+d+1}\mathbb{KO}(C^*(G))$$

is weakly homotopic to $\Omega^{\infty+1}\,\mathrm{topind}_G$.

Bonus: The action of the diffeomorphism group

Pulling back metrics yields an action

$$\mathrm{Diff}(M) \curvearrowright \mathcal{R}^+(M)$$

By comparing the two fibrations

$$\begin{tikzcd}
\mathcal{R}^+(M) \arrow[d] \\
\mathcal{R}^+(M) /\!\!/ \mathrm{Diff}(M) \arrow[d] \\
\mathrm{BDiff}(M)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{R}^+(M) \arrow[r, equal] \arrow[d] & \mathcal{R}^+(M) \arrow[d] \\
\mathcal{W}^{[3],\mathrm{psc}}_{\theta,\emptyset} \arrow[r, hook] \arrow[d] & \operatorname{hocolim}_{T \in \mathcal{K}^{[k]}} \mathcal{W}^{[3],\mathrm{psc}}_{\theta,T} \arrow[d] \\
\mathcal{W}^{[3]}_{\theta,\emptyset} \arrow[r, hook] & \operatorname{hocolim}_{T \in \mathcal{K}^{[k]}} \mathcal{W}^{[3]}_{\theta,T} \simeq \Omega^{\infty-1} MT\theta^{[3]}
\end{tikzcd}$$

one can prove the following ...

Theorem (B.)

Let M be a compact spin manifold of dimension $d \geqslant 5$ and $\theta\colon \mathrm{BSpin}(d) \times BG \to BO(d)$ its tangential 2-type. Then for $h \in \mathcal{R}^+(\partial M)$ the action map $\Sigma\colon \mathrm{Diff}_\partial^{\mathrm{Spin},G}(M) \to \mathrm{hAut}(\mathcal{R}^+(M)_h)$ factors up to homotopy through

$$\mathrm{Diff}_\partial^{\mathrm{Spin},G}(M) \simeq \Omega B\mathrm{Diff}_\partial^{\mathrm{Spin},G}(M) \xrightarrow{\Omega\alpha_M} \Omega^{\infty+1}\left(MT\mathrm{Spin}(d) \wedge BG_+\right)$$

Theorem (B.)

Let M be a compact manifold of dimension $d \geqslant 5$ with $(M, \partial M)$ 2-connected, θ its tangential 2-type and $h \in \mathcal{R}^+(\partial M)$. Then the action map $\Sigma_*\colon \pi_* \mathrm{Diff}_\partial(M) \to \pi_*(\mathrm{hAut}(\mathcal{R}^+(M)_h))$ factors through the map induced by

$$\mathrm{Diff}_\partial(M) \simeq \Omega B\mathrm{Diff}_\partial(M) \xrightarrow{\Omega\alpha_M} \Omega^{\infty+1} MT\theta(d)$$

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